Foundations of Physics, Vol. 32, No. 2, February 2002 (© 2002)

Book Review

Beyond Einstein's Velocity Addition Law. By Abraham A. Ungar. Fundamental Theories of Physics **117**. Kluwer Academic, Dordrecht, The Netherlands, 2001, xlii + 413 pp., $138.00 (hardcover).

By most standards, the basic four-dimensional spacetime formalism proposed by Hermann Minkowski in 1908 for use in special relativity has performed passably well. Spacetime diagrams in particular are ubiquitous to teaching and research alike. However, from the inception of the spacetime approach, and afterwards at an interval of roughly twenty years, rival formalisms have arisen with the intent either to complement the dominant technique or to supplant it entirely. The most recent entry in the lists is Abraham Ungar's gyrovector formalism, which after its discovery in 1988 underwent significant development by its author, culminating in the work under review. As with most new formal methods, it has little to recommend its study in the way of new physical results or insights, but may be compared with the spacetime approach in terms of the elegance of respective proofs of selected theorems. One might also compare Ungar's method to those of his immediate forerunners. On both counts, the gyroformalism proves to be worthy of physicists' attention.

When H. Minkowski began pondering the structure of the Lorentz group in 1907, one of the first things he noticed was that geometrical relations between velocity vectors measured in inertial frames of reference are not Euclidean (as in classical kinematics), but hyperbolic. The spacetime formalism he went on to develop for the physics of relativity, however, did not exploit this insight, rather, its reliance on an imaginary temporal coordinate tended to obscure the non-Euclidean nature of the Minkowski metric. Nonetheless, his system, developed by A. Sommerfeld and others into a full-blown vectorial analysis, and doted with a visually intuitive model in the form of a spacetime diagram, rose rapidly to dominate theoretical work in relativity. Mathematicians like V. Varičak and E. Borel then saw that by employing a *real* temporal coordinate, they could exploit hyperbolic trigonometry, and went on inaugurate a new, non-Euclidean style of relativity. This alternative style was largely neglected by contemporary physicists, who also ignored Borel's subsequent discovery of a kinematic phenomenon later known as Thomas precession. Even so, the non-Euclidean style found steady employment in relativity textbooks, where it was used to present velocity composition.

Over the years, there have been a handful of attempts to promote the non-Euclidean style for use in problem solving in relativity and electrodynamics, the failure of which to attract any substantial following, compounded by the absence of any positive results must give pause to anyone considering a similar undertaking. Until recently, no one was in a position to offer an improvement on the tools available since 1912. In his new book, Ungar furnishes the crucial missing element from the panoply of the non-Euclidean style: an elegant nonassociative algebraic formalism that fully exploits the structure of Einstein's law of velocity composition. The formalism relies on what the author calls the "missing link" between Einstein's velocity addition formula and ordinary vector addition: Thomas precession, or the angular difference in relative velocity that results when one changes the order of frame velocities in the velocity addition formula, the magnitude of this relative velocity being invariant with respect to frame order.

Ungar lays out for the reader a sort of vector algebra in hyperbolic space, based on the notion of a gyrovector. A gyrovector space differs in general from a vector space in virtue of inclusion of Thomas precession, and exclusion of the vector distributive law. As a result, when expressed in terms of gyrovectors, Einstein's (noncommutative) velocity addition law becomes "gyrocommutative," in that the two possible expressions for relative velocity for two inertial frames in non-parallel motion are related by a Thomas precession. One advantage of this approach is that hyperbolic geometry segues into Euclidean geometry, with notions such as group, vector, and line passing over to their hyperbolic gyro-counterparts (gyrogroup, etc.). Ungar's book is devoted to the presentation of the theory of gyrogroups and gyrovector spaces derived from the formal wedding of velocity addition to Thomas precession.

One might suppose that there is a price to pay in mathematical regularity when replacing ordinary vector addition with Einstein's addition, but Ungar shows that the latter supports gyrocommutative and gyroassociative binary operations, in full analogy to the former. Likewise, some gyrocommutative and gyroassociative binary operations support scalar multiplication, giving rise to gyrovector spaces, which provide the setting for various models of hyperbolic geometry, just as vector spaces form the setting for the common model of Euclidean geometry. In particular, Einstein gyrovector spaces provide the setting for the Beltrami ball model of hyperbolic geometry, while Möbius gyrovector spaces provide the setting for the Poincaré ball model of hyperbolic geometry.

The book begins with a potted history of Thomas precession and its abstract counterpart, Thomas "gyration" (thus explaining the plethora of "gyro" prefixes). This chapter will be of most interest to physicists, as it presents the basic method of gyrovectors and their motivation in a nutshell. Subsequent chapters are more formal, laying out in definition-lemma-theorem style the theory of gyrogroups and gyrovector spaces, with reference to the Beltrami and Poincaré models of hyperbolic geometry, including a nice discussion of common and hyperbolic parallel transport, studied with the methods of nonassociative algebra rather than those of differential geometry. In the book's final chapters, one finds a parametrization of the abstract Lorentz boost by gyrovectors, and two further parametrizations of nonlinear, pseudo-Lorentz transformations.

A trip "Beyond the Einstein Addition Law" will require a grasp of the basics of group theory, as well as access to a computer algebra program. For the perplexed, the author points to a wealth of references on both elementary and advanced concepts. Further assistance is provided by over eighty figures, most of which employ either Beltrami or Poincaré discs. The figures do yeoman service in aiding the reader to follow the geometric arguments, the assimilation of which may be tested by working out the exercises that close each chapter. Unfortunately, the text suffers from inadequate editing, with unnecessary repetition between chapters. A self-congratulatory tone, and an irrelevant recapitulation of an ongoing priority dispute are also to be regretted. Perhaps in relation to the latter, the author refers again and again to his first publication on K-loops, when he could have saved space and time by simply including the paper in an appendix.

Minor flaws in presentation aside, what can be said of the epistemic value of Ungar's formalism? There are three areas susceptible to application of the gyrogroup method: physics, non-Euclidean geometry and abstract algebra. All three areas appear ready to benefit from the method, which simplifies certain calculations and exploits Euclidean space intuitions. But will these advantages bring about new insights, and new results? For abstract algebra and non-Euclidean geometry, the author summarizes several recent results, for example, a new model of hyperbolic geometry (the "Ungar model"), and the discovery of cohyperbolic geometry (in which the angle sum of a triangle is π, but triangle medians are not concurrent). For physics, however, one still wonders if a compact, Euclidean vector notation accompanied by Beltrami or Poincaré models will ever outperform tensor calculus aligned with spacetime diagrams. To argue the case for gyrogroup methods, Ungar revives the age-old question of the reality of the Lorentz–FitzGerald contraction (LFC). He applies his gyrogroup-theoretic method to the problem of the apparent form of a sphere in uniform motion, and finds that it is measured as a flattened ellipsoid by an observer at rest. According to Ungar, this result vindicates Einstein's stance on the appearance of a sphere in uniform motion against the well-known argument of R. Penrose (1959), which holds that such flattening is not photographically observable. Penrose, however, considers light propagating with finite velocity from the sphere to an ideal photographic plate at rest, where Ungar assumes infinite signal velocity, a difference which clearly moots the comparison. For the case considered by the author, a more fruitful analogy might be drawn with the three-dimensional, mental "snapshots" described by J. Synge in his 1955 textbook on special relativity. Given the form of an object in motion as judged by a comoving inertial observer, Synge's method determines the object form as judged by an observer at rest, via iterative Lorentz transformation of surface points, where Ungar's algebra recovers the entire form in one fell swoop. Although apodictic proof of the gyrogroup method's creative power of discovery in physics is still forthcoming, physicists will appreciate the new geometric view of Einstein's velocity addition law and Thomas precession. Announced as a likely companion text to both undergraduate and graduate courses in physics, non-Euclidean geometry and abstract algebra, the book stands to be of most profit to advanced students and researchers. It will be a useful addition to all research libraries.

Scott Walter
LPHS-Archives Poincaré (UMR 7117)
Université Nancy 2
23 Bd. Albert 1er
54015 Nancy Cedex, France