Hermann Minkowski's approach to physics

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Abstract Hermann Minkowski's contributions to relativity theory are extraordinary from a disciplinary standpoint, in that so few mathematicians in Wilhelmine Germany took an interest in theoretical physics. Based on correspondence, lectures, research notes and the accounts of his students and colleagues, Minkowski's interest in mechanics and physics is retraced, from his early career to his discovery of spacetime.

The theory of spacetime discovered by the Göttingen mathematician Hermann Minkowski (1864–1909) is known to many from his lecture "*Raum und Zeit,*" delivered at the annual meeting of the German Association for Natural Scientists and Physicians in Cologne, on 21 September 1908. As the first speaker in the first mathematics session of the congress, Minkowski argued famously that certain circumstances required scientists to discard the traditional view of physical space as an Euclidean three-space, in favor of a four-dimensional geometry characterized by the invariance of a certain quadratic form. Grandiloquent in style, Minkowski's lecture struck a chord among scientists and philosophers, and upon publication, generated a reaction that was phenomenal in terms of sheer publication numbers and disciplinary breadth.

Minkowski's Cologne lecture was not his first on the topic of relativity theory, and in fact the main results of "*Raum und Zeit*" are all found in the appendix to Minkowski's long, technically demanding paper on the electrodynamics of moving media, presented to the Göttingen Academy of Science in December 1907, and published in the *Göttinger Nachrichten* the following April [54]. The latter paper extended the reach of the principle of relativity to the electrodynamics of moving media, by reformulating the Maxwell equations in four-dimensional terms that guaranteed the covariance of expressions with respect to the Lorentz transformation. Reading Minkowski's paper was rendered difficult by his idiosyncratic formalism, and it was panned for this reason by one of the founders of relativity theory, the young Albert Einstein, along with his co-author Jakob Laub. Einstein and Laub felt Minkowski's four-dimensional notation placed "rather great demands" on the reader [19, p. 532], and for this reason, they stripped it from his theory, replacing it with standard vector notation.

Spacetime had just been summarily dismissed in the pages of Germany's leading journal of theoretical physics, by two of Minkowski's former students. One imagines that this turn of events gave him pause, and it certainly seems it did, since the approach to relativity that Minkowski adopted in "*Raum und Zeit*" precluded any such tampering, by presenting what he called a "purely mathematical" argument leading to his spacetime theory, devoid of physical considerations [55, p. 75]. This had the effect of directing attention to the mathematics that physicists seemed bent on suppressing. A century after the fact, it goes without saying that Minkowski's strategy met with success.

The reasons for this success merit further study. The reception of "*Raum und Zeit*" itself was a complex affair, as the same lecture that was hailed by mathematicians, managed to scandalize physicists [89, 90]. What was at stake was not so much a matter of priority of discovery, as that of the role of mathematics in understanding the electrodynamics of moving matter, or even physical phenomena in general. Minkowski's views on the latter question were distinct not only from those of physicists, but also from those of almost all mathematicians in germanophone universities, who simply ignored physics.[1]

If his acute interest in questions of physics is known to have set Minkowski apart from his contemporaries in mathematics, next to nothing has been written to characterize this interest, or to investigate its sources. In what follows, we will see that although Minkowski published almost nothing in the domain of analytical mechanics, his interest in this subject spans the better part of his career, and segues into his later reformulation of electrodynamics. Similarly, while Minkowski published nothing on heat radiation, his unpublished 1907 lectures on this subject provide a link to Max Planck's relativistic thermodynamics, and to Einstein's theory of relativity.

The present account of Minkowski's approach to physics follows his career path from Bonn to Königsberg and Zürich, and finally to Göttingen, and is divided into three sections. In the first section, Minkowski's formal education is summarized, and his interest in mechanics is characterized based on correspondence and manuscript lecture notes, until the time of his arrival in Göttingen in 1902. The second section covers Minkowski's activities in mechanics and physics from 1902 until his lectures on heat radiation in 1907, while the final section treats the latter lectures, and Minkowski's contributions to relativity theory.

1 From Alexoten to Zürich

Hermann Minkowski (1864–1909) was born in the hamlet of Alexoten (Alexotas), Russia (near Kaunas in modern-day Lithuania), the fifth child in a family of six. The oldest son, Max, was sent away to Prussia for schooling, because of limited educational opportunities available to Jews in the Russian empire under Tsar Alexander II. Young Hermann was home-schooled until the age of seven, when he was enrolled in a school in Kaunas [73, p. 325]. A year later, the Minkowski family emigrated to Königsberg, Germany (now Kaliningrad, Russia), where Hermann was naturalized in 1872 [77, p. 9]. He attended the Altstädtisches Gymnasium with Willy Wien, while Max Wien and Arnold Sommerfeld were in the junior classes; all three went on to lead distinguished careers in physics [86]. Hermann began university studies in Königsberg in April 1880, hearing lectures by Heinrich Weber, and by Franz Neumann's last doctoral student, Woldemar Voigt. In 1882, Minkowski submitted a paper [48] for the Paris Academy's *Grand prix des sciences mathématiques*, which invited contributions to the theory of the decomposition of integers into five squares. Minkowski shared the prize with the seasoned British mathematician H.J.S. Smith, who had actually provided a solution to the decomposition problem in 1867 [82].

[1] Jungnickel and McCormmach [28, vol. 1, p. 185] identify Minkowski as one of the very few mathematicians who contributed to physics in Wilhelmine Germany, along with Carl Neumann and David Hilbert.

From the winter semester (WS) of 1882, Minkowski spent three terms in Berlin, following the lectures of Kummer, Kronecker, Runge, Weierstrass, Helmholtz, and Kirchhoff [80, p. 76]. He returned to Königsberg, where he and his friend David Hilbert (1862–1943) both heard lectures by Ferdinand Lindemann (who had replaced Weber), and by Adolf Hurwitz (1859–1919). Under Lindemann's direction, Minkowski defended a thesis on quadratic forms in 1885 [47], and after obligatory military service, habilitated in Bonn in 1887. The *Probevorlesung* he delivered on this occasion, "On the history of probability calculus", disclosed some of the premises of his later work on the geometry of numbers [80, 81].

In Bonn, Minkowski took up a problem of mathematical physics studied by W. Thomson, Kirchhoff, Clebsch, and others: to find the motion of solids immersed in a perfect liquid. For the case of force-free motion, Minkowski devised a method applicable to a solid of any form, which Helmholtz agreed to communicate to the Berlin Academy of Science [49]. Along with fluid mechanics, Minkowski studied number theory and the theory of elasticity, including a paper in the latter field by Voigt, now a professor of theoretical physics in Göttingen, one of only two such chairs in Germany.[2] Minkowski's encounter with Voigt's work was an unpleasant one. Voigt's theoretical study of elasticity "really horrified" him; it was "utterly incomprehensible," he wrote to Hilbert, "how anyone can apply odd calculations in the hope that later, perhaps, someone will be found who can get something out of it."[3]

The following year, Minkowski delved further into physics, confessing to Hilbert that he was "now swimming almost entirely in a physical channel," and spending time in Bonn's Institute of Physics, directed by Heinrich Hertz (1857–1894).[4] Hertz was then at the height of his fame for discovering wireless electromagnetic waves, and his laboratory acted as a magnet for talented young physicists. Just how much contact Minkowski had with Hertz, or with the other physicists in attendance, including Philipp Lenard, Vilhelm Bjerknes, and Kristian Birkeland, is not clear. When Minkowski showed up for a laboratory course, Hertz noted this in his diary; he also invited the young mathematician over for dinner, which was an occasion for them to work together on problems of physics [20, p. 462].

The collaboration with Hertz went no further it seems, but the encounter made a strong impression on Minkowski. Hertz fell ill, and died on New Year's Day, 1894; had he lived, Minkowski speculated later to Hilbert, he would have taken more of an interest in physics at the time [26, p. 462]. Much like Hertz, who worked out new principles of mechanics in the early 1890s [25], Minkowski cultivated his interest in theoretical mechanics while in Bonn, reviewing twenty articles on rigid-body dynamics and potential theory for the abstract journal *Jahrbuch über die Fortschritte der Mathematik*, including articles by Kirchhoff, Sophie Kovalevski, Carl Neumann, and Poincaré, along with fifty-five papers on number theory.

It was also in the early 1890s that Minkowski made his mark in number theory, publishing memoirs that formed the basis of a new mathematical

[2] The other chair was held by Kirchhoff in Berlin.

[3] Voigt [87]; H. Minkowski, Untitled notebook, Arc. 4°1712, Jewish National and University Library (JNUL); Minkowski to Hilbert, 19.06.1889 [77, p. 36]. Cf. Pyenson [72, p. 64], where the translation differs.

[4] Minkowski to Hilbert, 22.12.1890 [77, p. 39].

sub-discipline: the geometry of numbers [22]. Following Gauss and Dirichlet's appeal to spatial intuition for research on quadratic forms in the early nineteenth century, and Hermite's work of 1850 on the reduction of quadratic forms, Minkowski published a series of related results culminating in the seminal *Geometrie der Zahlen* [50].

Promoted to the rank of associate professor in 1892, Minkowski quit Bonn two years later for a position of equal rank in Königsberg, thereby rejoining Hilbert. The next year, however, Hilbert was hired away by Göttingen, making way for Minkowski's promotion to full professor [78, pp. 52, 197]. Having wrapped up his book on the geometry of numbers, Minkowski returned to mathematical physics, a field in which earlier, as he explained to Hurwitz, he had been "greatly interested," even if he had "nothing in this field ready for publication" [35, p. 230].

Five months after writing to Hurwitz of his return to mathematical physics, Minkowski joined his former teacher on the faculty of Zürich Polytechnic (which later became the ETH). The move offered him a substantial salary increase, and exposure to students preparing careers in the practical arts of engineering, as well as in mathematics. During his six years on the Polytechnic faculty, he lectured on analytical mechanics, hydrodynamics, potential theory, and variational calculus, in addition to number theory, the theory of functions, algebra, and partial differential equations.[5]

While Minkowski was not in the habit of writing out his lectures in detail, in Zürich he began to think about publishing a monograph and encyclopedia entries on mechanics. In 1899, he considered writing an article on hydrodynamics for Felix Klein's monumental *Encyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen* (*EMW*), published by B.G. Teubner, as well as one on capillarity. These topics belonged in two separate volumes: one on mechanics, edited by Klein and his former student Conrad Müller, the other on physics, edited by Sommerfeld. The dividing line between the two was clear to Minkowski, who promised Sommerfeld that if he ended up writing for Klein's volume, he would "cede to mechanics no questions belonging in your volume."[6] As it turned out, Klein, during a trip to Great Britain and Ireland with Sommerfeld in August, met with A.E.H. Love, the new Sedleian Chair in Natural Philosophy at Oxford, to discuss the possibility of contributing an article on hydrodynamics.[7] Minkowski contacted Love in the fall about an eventual contribution,[8] but in the end, Love wrote the two entries on hydrodynamics in the *EMW* [42, 43].

With Love set to cover hydrodynamics in the *EMW*, Minkowski turned his attention to capillarity, a topic he addressed in lectures on mechanics in WS 1897, WS 1899, and SS 1900 at Zürich Polytechnic.[9] Like many other *EMW* authors, Minkowski was encouraged by the publishing house of B.G. Teubner to write a textbook building upon his entry, to appear in a new tie-in

[5] See [85, Doc. 28]; H. Minkowski papers, JNUL.

[6] Minkowski to Sommerfeld, 18.11.1899, Archiv HS 1977-28/A 233, Deutsches Museum München. Sommerfeld had taken on the task of editing the physics volume in the summer of 1898 [16, vol. 1, p. 40], and probably commissioned Minkowski to write an article on capillarity shortly thereafter.

[7] Klein to W. Dyck, 20.08.1899 [16, vol. 1, p. 43].

[8] Minkowski to Sommerfeld, 18.11.1899.

[9] H. Minkowski papers, JNUL. Beginning in December 1899, Minkowski's notebooks show calculations and bibliographic references pertaining to capillarity.

collection entitled *"B.G. Teubners Sammlung von Lehrbüchern auf dem Gebiete der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen."* Minkowski put off the commission, explaining to the publisher that he intended to write such a textbook, but could not commit himself to do so before making "a final decision on the extent of his subject matter."[10] Apparently, Minkowski was at liberty to define the limits of his *EMW* entry. The decision before him was not a simple one, due in part to the broad range of capillary phenomena, but also to the absence of consensus among theorists on how to approach the subject.

Minkowski's course notes from Zürich Polytechnic bear witness to his familiarity with a wide array of textbooks on mechanics by French, British, Danish, and German authors. When the first volumes of Klein and Sommerfeld's *Theorie des Kreisels* [31] appeared, Minkowski wrote Sommerfeld that the momentum-based approach of this work appealed to him, and was in fact the only one he had ever employed in his own lectures on the basic equations of mechanics.[11] In reality, Minkowski was deeply annoyed by Klein's neglect of his 1888 study of the motion of solids immersed in fluid flows ([49], mentioned above), and complained to Hilbert a few weeks later that there was much in this paper that Klein and Sommerfeld could have used in their textbook, "if only they had read it."[12]

Finding students willing to take his courses in Zürich turned out to be almost as difficult as finding readers for his paper on hydrodynamics. Among the handful of students who attended Minkowski's lectures on mechanics, however, was Albert Einstein. According to a fellow student's recollection, the young Einstein was enthused by Minkowski's approach to capillarity, remarking after a lecture on this topic, "That is the first lecture on mathematical physics we have heard at Poly" [34, p. 21]. Minkowski did not pick out Einstein as a particularly promising prospect, but in fact the only student from Polytechnic he mentioned in correspondence with Hilbert was Walter Ritz (1878–1909), a 1901 graduate who wrote a doctoral thesis on atomic spectra in Göttingen under Voigt's supervision.[13]

[10] B.G. Teubner publishing house to H. Minkowski, 10.01.1901, with draft reply on verso, H. Minkowski papers, JNUL.

[11] Minkowski to Sommerfeld, 30.10.1898, MSS 1013 A, Smithsonian Institution, Dibner Library.

[12] Minkowski to Hilbert, 06.12.1898 [77, pp. 109–111].

[13] Minkowski to Hilbert, 11.03.1901 [77, pp. 138–139]. On Ritz's Ph.D. thesis, see [11]. Ritz later developed the Rydberg-Ritz combination principle, and an emission theory of electrodynamics [45]. His correspondence with Paul Ehrenfest (Boerhaave Museum) tells of Minkowski's encouragement of his study of blackbody radiation [74]; Minkowski also approved Ritz's habilitation in physics, as member of a jury including Hilbert, Voigt and Runge [92].

Fig. 1 Hermann Minkowski [27, vol. 1]

2 Göttingen I: Mechanics and electrons

Although he had wanted to teach at the University of Zürich, Minkowski was not allowed to do so, and had to content himself with his classes at Polytechnic.[14] Zürich Polytechnic was the training ground for the Swiss technical elite, but for Minkowski it remained a school from which "a complete knowledge of mathematics could not be obtained," as he remarked later in an evaluation of Einstein's theory of relativity.[15] An opportunity to return to university

[14]Minkowski to Hilbert, 05.09.1896 [77, p. 85].

[15]Undated manuscript, ca. April 1908, Math. Archiv 60:4, 52, Handschriftenabteilung, Niedersächsische Staats- und Universitätsbibliothek (NSUB).

instruction arrived in 1902, when the University of Göttingen tendered him a chair – created on his behalf – in pure mathematics. Göttingen's extraordinary offer came about after Hilbert was offered Lazarus Fuchs' chair in Berlin. Rather than move north, Hilbert decided he would be better off in Göttingen, if only Minkowski were allowed to join him and Klein on the mathematics faculty [75, p. 436]. The deal was made, and Hilbert got his way, to the satisfaction of both Minkowski and Klein. A photograph (Fig. 1) taken in Zürich shows him in stiff collar and tie.

Once in Göttingen, Minkowski continued to study the theory of capillarity, working this topic into a course on minimal surfaces in the summer of 1903. He prepared a draft of his *EMW* article in 1904, but it was not until September 1906 that a set of corrected proofs reached the publisher, in a version that required further revision, as we will see later.[16]

While maintaining his earlier interest in capillarity, Minkowski sampled familiar topics in mechanics, as well as others in physics that were new to him. In 1903, for example, he co-directed a seminar on stability with Hilbert, highlighting several approaches, including that of Klein and Sommerfeld in the second volume of their textbook, and those published in *Crelle* by Kneser, Lyapunov, and Levi-Civita. The stability seminar also took up Love's *Mathematical Theory of Elasticity* [41], and Poincaré's *Mécanique céleste* [61] and *Figures d'équilibre d'une masse fluide* [66], among other works.[17] The following summer, the Hilbert-Minkowski duo led a seminar on mechanics, attended by Max Born and Paul Ehrenfest [84, p. 152]. One year later, in SS 1905, the duo co-directed a third seminar, this time targeting a new area of physics, to which several Göttingen scientists had contributed groundbreaking results: electron theory.

The theory of the electron was an unusual choice of research topic, for a couple of reasons. First of all, neither Hilbert nor Minkowski had ever lectured (or published) on optics or electromagnetism. In the second place, the physics of the electron was not a research concern of Göttingen's professor of theoretical physics, Woldemar Voigt. In fact, Voigt was an outspoken critic of electron theory, which he felt led only to "exquisite complexities" [88, p. 362]. Hilbert and Minkowski could have ignored Voigt's view, but instead, they chose to confront it directly. Voigt's "neutral" theory of the Zeeman effect had been taken up by his former doctoral student Paul Drude, in his *Theory of Optics* [15], so the mathematicians put Drude's treatment of the Zeeman effect (i.e., a certain splitting of spectral lines in a strong magnetic field) on the syllabus of the electron-theory seminar. It was scheduled for study in the seminar's final session, alongside Poincaré's presentation [65, p. 544] of Lorentz's Nobel Prize-winning explanation – based on electron theory – of the same phenomenon. The seminar participants were encouraged thereby to compare Lorentz's microscopic approach to the Zeeman effect with its less-sophisticated phenomenological rival, devised by Göttingen's senior theoretical physicist.

While intellectual affiliation and local bias argued against an electron-

[16]Notebooks "Vorlesungen über Minimalflächen" and "Capillarität," H. Minkowski papers, JNUL; Minkowski to Sommerfeld, 01.09.1906, Archiv HS 1977-28/A 233, Deutsches Museum München.

[17]Syllabus, Minkowski papers, JNUL; Hilbert Nachlass 570/1, NSUB. I thank Tilman Sauer for pointing out Hilbert's copy of the syllabus, and providing a transcription.

theory seminar, several factors rendered this an attractive choice for Hilbert and Minkowski in 1905 [12, § 3.2]. First and foremost among these is the fact already mentioned: many of the leading lights of research in this domain were on hand in Göttingen, making electron theory a natural seminar subject. The resident electron theorists included Emil Wiechert, director of the Geophysical Institute and co-discoverer (with J.J. Thomson) of the electron; Max Abraham, Privatdozent and former student of Max Planck; Karl Schwarzschild, director of the astronomical observatory; Gustav Herglotz, Privatdozent in mathematics and astronomy; and Paul Hertz, who wrote a doctoral thesis on electron motion with Abraham's counsel, officially under Hilbert's supervision. Other experts in electron physics not resident in Göttingen, but with strong ties to the Göttingen community included Arnold Sommerfeld, a former Göttingen Privatdozent in mathematics, who had published several lengthy studies of electron motion, and Walter Kaufmann, a Privatdozent in physics who performed cathode-ray deflection experiments in Göttingen until his recruitment in 1903 by the University of Bonn.

A second factor in the decision to run an electron-theory seminar was a general sense that this was a problem domain with potential for development, that lent itself to mathematical analysis. As a research domain, electron theory was quite new, dating only from the early 1890s, and gaining significant momentum with the discovery of the electron in 1897 [10, p. 173]. It was by no means a stable field, amenable to the sort of axiomatic foundation that Hilbert envisaged for theories of physics, as problem number six in his famous list of twenty-three worthy problems [23]. In 1903, Hilbert did not even feel that continuum mechanics was ready for an axiomatic treatment [12, p. 128]. The rapid development of electron theory at this time argued against an axiomatic approach, while offering a series of fresh problems for analysis.

A third factor in choosing to run a seminar on electron theory was this topic's foundational importance for physics in general. For those physicists who felt that all physical processes lent themselves to a microphysical reduction based on interactions between electrons and the ether, the potential significance of electron theory was understood to be quite profound. Just such a theory was announced by Willy Wien in December 1900, and referred to as the "electromagnetic worldview," because the basic laws of interaction were assumed to be similar in form to those governing the electromagnetic field in Maxwell's theory [46, p. 477]. Wien, who directed the Würzburg Physical Institute, and was on good terms with his old Königsberg friends Hilbert and Minkowski, was invited by the *Mathematiker Vereinigung* to lecture on the partial differential equations of physics at the German Association's annual meeting in Meran, less than a month after the close of the electron-theory seminar. Wien's talk in Meran outlined some of the more difficult problems facing physicists, including one from electron theory: the insolubility of Sommerfeld's force-free equation of motion of a rigid spherical electron. Solving such outstanding problems, Wien argued, required "more extensive collaboration" between physicists and mathematicians [93].

Of a like mind with Wien on this subject, Minkowski felt collaboration between mathematicians and physicists was required for progress in physics. Collaboration between physicists and astronomers, as he remarked in a commemorative lecture on Dirichlet to the Göttingen mathematical society just a few months before the electron-theory seminar, "would bring forth no essentially new elements." The new elements necessary for progress in physics could be provided only by mathematicians, in Minkowski's view. The mathematician who joined up with a physicist or an astronomer, Minkowski continued, could "unfold all the resources of his science and the results of his own research," and obtain "drive and stimulation for investigations of new areas of mathematical problems" [51, p. 162]. Such interdisciplinary collaboration was not limited to specialists in analytical mechanics. Pointing to Fourier series and Fresnel diffraction for instance, Minkowski suggested that even number theorists could contribute to progress in physics and astronomy [51, p. 155].

Greater collaboration between physicists and mathematicians was also an essential part of Felix Klein's grand scheme to revitalize the exact sciences in Göttingen. From the early 1890s, with the support of an influential friend in the Prussian Ministry of Culture, Friedrich Althoff, and funding from both the government and local industrialists, Klein facilitated the construction of scientific institutes directed by the likes of Emil Wiechert (geophysics), and Ludwig Prandtl (technical physics), and succeeded in attracting other highly-talented academics to Göttingen, including Hilbert, Schwarzschild, Minkowski and Runge. Klein's and Hilbert's leadership contributed powerfully to the rise of Göttingen as a scientific center, as shown in detailed histories by Manegold [44] and Rowe [76].

On a somewhat smaller scale, the electron-theory seminar represented another means for collaboration between mathematicians and physicists. Wiechert and Herglotz appear both to have participated by taking turns with Hilbert and Minkowski in leading the seminar. Schwarzschild, Abraham, and Hertz, on the other hand, appear not to have shouldered any particular responsibility in the seminar's organization, and it is not clear that they participated at all in the weekly discussions. Several students were given tasks to perform at one session of the seminar or another, perhaps as presenters, or note-takers. Among these students were two who later established their scientific credentials in the domain of Minkowskian relativity: Max Laue and Max Born.[18]

Max Born began his studies in his home town of Breslau (now Wrocław, Poland), moving on to Heidelberg and then Zürich, arriving in the latter city just after Minkowski's departure. Born's stepmother was acquainted with Minkowski from Königsberg, and to smooth his arrival in Göttingen, she provided young Max with a letter of introduction [7]. It was Born who was designated the official notetaker for Minkowski's SS 1904 course on affine geometry,[19] and Born again who was paired with Minkowski in week three of the electron-theory seminar. He later became Minkowski's assistant, although the position lasted only a few weeks, due to the sudden death of his erstwhile mentor.

According to Born's recollections of the electron-theory seminar, written half a century or so later, Minkowski "occasionally hinted" that he was engaged with what would later be known as the Lorentz transformation, and provided an "inkling" of the results he communicated in 1908 [5, p. 245]. In a 1962 interview, Born went a bit further, recalling how "Minkowski's first ideas about

[18] Both Laue and Born went on to win the Nobel Prize in physics, in 1914 and 1954, respectively.

[19] H. Minkowski, Vorlesungen über Linien- und Kugelgeometrie, edited by Max Born, Nachlass Born 1808, Staatsbibliothek Preußischer Kulturbesitz, Berlin.

relativity were already worked out and shown" in the electron-theory seminar [7].

It is unclear from Born's recollections just what Minkowski disclosed in the summer seminar of 1905, before he had read the first papers on relativity theory by Poincaré [68] and Einstein [18]. In the 1940s, Born remembered hearing Minkowski tell of his first encounter with Einstein's theory [8, p. 131]:

> [Minkowski] told me later that it came to him as a great shock when Einstein published his paper in which the equivalence of the different local times of observers was pronounced; for he had reached the same conclusions independently but did not wish to publish them because he wished first to work out the mathematical structure in all its splendor.

This story of Minkowski's recollection of his encounter with Einstein's paper on relativity is curious, in that the idea of the observable equivalence of clocks in uniform motion had been broached by Poincaré in one of the papers studied during the first session of the electron-theory seminar. It is possible, of course, that Poincaré's operational definition of local time escaped Minkowski's attention, or that Minkowski was thinking of an *exact* equivalence of timekeepers.[20]

The electron-theory seminar syllabus featured a total of ten papers on different aspects of electron physics by Göttingen theorists, plus three works that formed the backbone of the seminar: Lorentz's *Versuch einen Theorie der elektrischen und optischen Erscheinungen in bewegten Körpern* [37], Poincaré's *Électricité et optique* [65], and Lorentz's *EMW* entry on electron theory [40].[21] It is not clear from the archival syllabus whether Hertz's macroscopic theory of the electrodynamics of moving bodies [24] was discussed during the seminar, although week three was devoted to the optics of moving systems, and was based in part on Poincaré's treatise, which gives an overview of Hertz's theory. The nine-week seminar, which ran to 31 July 1905, covered the principal works on electron theory up to 1904, in accordance with Lorentz's *EMW* review. As Pyenson observes [72], the syllabus neglected several topics covered by Lorentz's comprehensive review, as well as subsequently-published works. Most notably among the latter, the syllabus omitted Lorentz's article in the Amsterdam *Proceedings* [38], containing what Poincaré later named the "Lorentz transformation," as well as Poincaré's address to the scientific congress at the World's Fair in Saint-Louis, which placed the principle of relativity at the base of a new mechanics, for which the speed of light is a limiting velocity [67].

Poincaré's Saint-Louis lecture was familiar to Göttingen's mathematicians, having been presented by Conrad Müller in the 24 January 1905 meeting of the Göttingen mathematical society. Members of the mathematical society, in line with their particular area of expertise or interest, reported regularly on recent work in mathematics and theoretical physics. Aspects of electron theory, for example, were addressed in this venue by Abraham, Paul Hertz,

[20] Poincaré [64] asserted that Lorentz's local time was given by light-synchronized clocks at relative rest in an inertial frame of reference, to first order of approximation in v/c (or frame velocity with respect to the ether, divided by the speed of light). On Poincaré's definition of local time, see Darrigol [13, p. 28].

[21] Hilbert Nachlass 570/9, NSUB; facsimile in [72, p. 57].

and Herglotz.[22]

One of Minkowski's areas of expertise was the science of Henri Poincaré. Out of the twenty-three talks Minkowski delivered to the Göttingen mathematical society from 1902 to 1909, three concerned Poincaré's work in mathematics, and two talks dealt with his contributions to physics. In all, Minkowski gave six talks to the mathematical society on questions of physics, five of which came after the electron-theory seminar. These talks, in chronological order, were as follows: (1) on his *EMW* entry on capillarity (23.06.1903); (2) on Poincaré's lectures on capillarity [62] (06.02.1906); (3) on a recent paper by Minkowski's former Göttingen colleague Walther Nernst (on 26.06.1906), in which Nernst proved the Heat Theorem; (4) on the theories of radiation of Lorentz, Rayleigh, W. Wien and Planck (11.12.1906); and (5–6) on the equations of electrodynamics (05.11.1907 and 28.07.1908), where Minkowski presented his ideas on relativity theory.[23] The next section shows how the progression of topics chosen by Minkowski for presentation to the Göttingen mathematical society relates to his growing familiarity with contemporary electrodynamics.

3 Göttingen II: From black-body theory to spacetime

The theory of heat radiation was not among the topics addressed during the 1905 electron-theory seminar, although it was covered by one of the textbooks on the seminar syllabus, Drude's *Theory of Optics* [15]. More than likely, heat radiation was the subject Minkowski meant to address at the 78th meeting of the German Association in Stuttgart, "on a to-be-determined topic of theoretical physics," but did not deliver.[24] This was the topic, in any case, of his lecture to the Göttingen mathematical society in December of that same year, as mentioned above.

In addition to the study of radiation theory, Minkowski pursued two publication projects in the wake of the electron-theory seminar: an introductory text on number theory [52], and his entry on capillarity for the *EMW*. The early weeks of 1907 were taken up with making corrections to the latter article, following a series of critical remarks by H. Kamerlingh Onnes and Sommerfeld. By this time, Minkowski had studied the theory of capillarity for nearly a decade. Having left the details of the thermodynamic theory of capillarity to Kamerlingh Onnes [29], Minkowski was quite ready to turn to other topics, remarking as he closed a letter to his editor detailing the final revisions of his review, "Nun, künftig interessantere Dinge."[25]

Following the electron-theory seminar, Minkowski's university lectures did not venture into the physical realm until the summer semester of 1907, when

[22] For a partial list of reports on questions of physics presented to the Göttingen mathematical society between 1903 and 1922 see Corry [12, pp. 454–457].

[23] *Jahresbericht der deutschen Mathematiker-Vereinigung* 12: 445; 15: 155, 407; 16:78; 17 (Mitteilungen und Nachrichten): 4, 85.

[24] Minkowski's talk was to be part of a session in the mathematics section, on 18 September 1906 (*Jahresbericht der deutschen Mathematiker-Vereinigung* 15, p. 446).

[25] Minkowski to Sommerfeld, 18.01.1907, Archiv HS 1977-28/A 233, Deutsches Museum München.

he gave his first-ever course on heat radiation.[26] Such an offering was atypical for a professor of pure mathematics, prompting Minkowski to explain in his first lecture:

> In this course I address not only physicists, but to an even greater extent pure mathematicians, otherwise inclined to remain more or less aloof from these subjects. It is my intention, and also [that of] Professor Hilbert, [who] thinks similarly about this, and has similar endeavors, to win over pure mathematicians [..] to the rich suggestions that flow into mathematics from the side of physics.[27]

According to the latter remark, Minkowski's and Hilbert's excursions into physics were naturally for the benefit of physicists, but performed with the hope that they would bring about a renewed engagement with problems of physics on the part of pure mathematicians, and thereby enrich the field of mathematics.

Although Minkowski did not say as much in his inaugural lecture on heat radiation, he saw this as a two-way street: just as mathematicians can benefit from the study of physics, physicists can benefit from the study of pure mathematics. Convincing either pure mathematicians or physicists of the truth of such a proposition was an uphill battle, to say the least. However, Minkowski succeeded in transferring his interest in number theory to at least one attentive physics student, Max Born, who used his new knowledge to produce results in statistical mechanics [6].

The notes Minkowski kept of his lectures on heat radiation offer a glimpse of his interests in this field. These notes are incomplete, jumbled and fairly cryptic, such that a partial reconstruction of the course's structure and content is the best we can hope to achieve. His theoretical outlook, however, was announced in the inaugural lecture. For Minkowski, the "essence [of heat radiation] is naturally and very precisely explained as an electromagnetic process." By the summer of 1907, a few young physicists, including Einstein, Ehrenfest, and Laue, had come to doubt that black-body radiation, or rather, Planck's distribution law, could ever be explained from the standpoint of Maxwell's theory of the electromagnetic field [36, p. 188].[28] Minkowski, however, was not inclined to doubt the electromagnetic foundations of black-body radiation. At issue for him were the hypotheses upon which Planck built his law, which were "inappropriate," and "difficult to make one's own."[29] Minkowski's notes do not reveal which of Planck's hypotheses were unacceptable, but at the time of Minkowski's lectures, the assumption of equiprobable states was often targeted by critics [36, p. 134ff].

In addition to the foundations of Planck's law, Minkowski's introductory

[26] *Physikalische Zeitschrift* 8, p. 63. During SS 1907, Minkowski also lectured on automorphic functions (with Klein and Hilbert), and variational calculus.

[27] H. Minkowski, Vorlesungen über die Theorie der Wärmestrahlung, Cod. Ms. Hilbert 707, p. 2, NSUB. Cf. [79], where the translation differs slightly.

[28] Minkowski might have learned of this contrary view directly from Ehrenfest, who was then in Göttingen with his wife Tatyana Afanasyeva. Both Paul and Tatyana had participated in Hilbert-Minkowski seminars, and had begun work on their influential *EMW* article on the conceptual foundations of statistical mechanics [17]. On the Ehrenfests' activity in Göttingen, see [32, p. 75ff]. Tatyana's participation in the SS 1903 Hilbert-Minkowski stability seminar is attested by both professors' copies of the syllabus (op. cit., note 17).

[29] Minkowski, Wärmestrahlung, p. 21.

lecture focused on the mechanics of radiation pressure, including the determination of the pressure of radiation reflected from a moving mirror. Minkowski chose this topic because it could be explained "without detailed calculations," in keeping with his wish to give his first lecture a "more popular character." From a historical standpoint, his choice of topic is of interest, as it reflects upon his engagement with relativity theory in the months leading up to his 5 November lecture to the Göttingen mathematical society, when this engagement was made public.

The problem of determining the pressure of radiation reflected from a moving mirror had been treated by Max Abraham [4, vol. 2, § 40] according to Huyghens' principle, a solution Minkowski found to be "mathematically the most convenient."[30] The problem studied by Abraham was an interesting one for Einstein, as well. Einstein probably knew of the lengthy solution to the moving-mirror problem Abraham published in the *Annalen der Physik* [1], but he did not mention it in his relativity paper of 1905. For Einstein, the formulas of his own work did not need to be underlined, and in fact, these agreed with Abraham's. Rather, Einstein insisted upon the broad reach of his general approach [18, p. 915]:

> All problems in the optics of moving bodies can be solved by the method employed here. The essential point is that the electric and magnetic force of light governed by a moving body is transformed to a coordinate system at rest relative to the body.

What are we to make of Minkowski's neglect of Einstein's solution to the moving-mirror problem? Did this neglect reflect his ignorance of Einstein's paper on relativity? This possibility can not be ruled out, since Einstein is absent from Minkowski's writings, up to and including the SS 1907 course notes on heat radiation. Alternatively, Minkowski may have been aware of Einstein's solution to the moving-mirror problem, but preferred that of Abraham. Einstein's theory of relativity was still quite new, and had yet to gain the confidence of theoretical physicists. Even Einstein's most influential supporter at this time, Max Planck found Abraham's approach to the moving-mirror problem to be superior to Einstein's. Planck wrote in his widely-read textbook on heat radiation that Abraham's "complete solution" of the moving-mirror problem represented a "foundation for the laws of the electrodynamics of moving bodies" [59, p. 71]. As for Einstein's solution, Planck simply ignored it, both in his textbook on heat radiation, and in his groundbreaking paper on relativistic thermodynamics [60]. It is quite telling that Minkowski referred to both of these publications in the notes to his SS 1907 course on heat radiation, and at the same time, he ignored Einstein's relativistic solution to the moving-mirror problem in favor of Abraham's non-relativistic method. Taken together, these two facts suggest that Minkowski, like Planck, did not yet appreciate the advantages afforded by Einstein's approach to the electrodynamics of moving bodies.

While perusing Planck's article [60], Minkowski would have come across a reference to Einstein's relativity paper [18]. He also might have learned of Einstein's paper from his electron-theoretical colleagues in Göttingen, from Max

[30] Minkowski, Wärmestrahlung, p. 19. Minkowski went on to apply the method to a simple case.

Abraham, for instance, although according to Born and Laue, between these two there was never any close collaboration.[31] Relativity theory attracted only a handful of theorists in 1907, in light of the contrary results of Kaufmann's cathode-ray deflection experiments of 1905, and the availability of plausible theoretical alternatives for the electrodynamics of moving bodies [14, p. 386]. In this context, Planck's criticism of Kaufmann's experiments and his public support of Einstein's theory helped counterbalance the views of skeptics like Abraham and Sommerfeld. Planck's attribution to Einstein of a "more general interpretation" of Lorentz's principle of relativity did much to establish Einstein's intellectual credentials in theoretical physics. It may also have led Minkowski to ask his former student for an copy of his relativity paper, for study in his winter seminar with Hilbert in Göttingen.[32]

Göttingen's 1907–1908 winter semester featured yet another Hilbert-Minkowski seminar, this time on the "partial differential equations of physics" (*Physikalische Zeitschrift* 8:712). One of the subjects taken up by the two mathematicians was that of the electrodynamics of moving media. The two principal theories here were those of Emil Cohn and H.A. Lorentz, both of which were presented using the vector notation earlier employed by Lorentz in the *EMW*. From a retrospective viewpoint, this would have been an ideal spot to introduce Einstein's method, mentioned above, in which the fields of a moving frame are transformed to those of a frame at rest (and vice-versa).[33] Instead, the equations of electrodynamics were studied by Hilbert and Minkowski according to the standard (pre-relativistic) texts of Lorentz [39], and Abraham-Föppl [4].

If Minkowski knew at this time how to write Maxwell's equations in four-dimensional form, he did not share his method with the seminar participants, at least not at first. The student notes of this seminar preserved in the archives are incomplete, and no syllabus has been found. What the surviving notes tell us is that the focus of the first weeks of the seminar was not on the form of the equations of electrodynamics, but on the empirical validity of Cohn's and Lorentz's theories of moving media. Cohn's theory was judged to be inconsistent with observation, but the notes come to an end before any conclusion is drawn for Lorentz's theory.[34]

The Hilbert-Minkowski seminar on the partial differential equations of physics was Minkowski's last regularly-scheduled offering on a subject of physics at the University of Göttingen.[35] Minkowski was not averse, however, to discussing physics during his lectures on mathematics. While the seminar began with a pre-relativistic approach to the electrodynamics of moving media, the principle of relativity took pride of place in another course led by Minkowski that same term: the theory of functions of a complex variable. The principle of relativity of Lorentz and Planck, Minkowski announced in his first lecture, was a "new triumph of mathematics." The "world," he explained further, was

[31] The Born-Laue obituary of Abraham [9] mentions an "incompatibility of character and temperament" between Minkowski and Abraham.

[32] Minkowski to Einstein, 09.10.1907 [33, Doc. 62].

[33] Einstein's relativity paper did not feature in the seminar's bibliography, according to student notes in the Göttingen archives (Hilbert 570/5, NSUB).

[34] Both of these theories were later evaluated by Minkowski from the standpoint of their compatibility with the principle of relativity [54, §§ 9, 10].

[35] The next two Hilbert-Minkowski seminars dealt with the principles of mathematics (SS 1908 and WS 1908); see *Physikalische Zeitschrift* 9:280, 688.

a “non-Euclidean manifold of 4 dimensions.”[36] Neither Einstein nor Poincaré were mentioned by Minkowski in this lecture, although both names featured in a lecture delivered to the Göttingen mathematical society a short while later, on 5 November.

In his lecture to the mathematical society, Minkowski developed his signature four-dimensional approach to relativity. This approach formed part of a new program: to reformulate the laws of physics in four-dimensional terms, based on the Lorentz-invariance of the quadratic form $x^2 + y^2 + z^2 - c^2t^2$, where x, y, z, are rectangular space coordinates, fixed “in ether,” t is time, and c is the vacuum speed of light [56, p. 374].

Minkowski acknowledged a step taken in the direction of such a reformulation by Poincaré, who had derived a Lorentz-covariant version of Newton’s law of gravitation [69, § 9]. He borrowed Poincaré’s definitions of position and force density with respect to a four-dimensional vector space where one axis is imaginary, and added a four-current density ϱ and a four-potential, ψ, with which he expressed Maxwell’s vacuum equations in the succinct form:

$$\Box\psi_j = -\varrho_j \qquad (j = 1, 2, 3, 4).$$

The possibility of expressing the equations of electrodynamics in this way was something “not even Poincaré” had seen.[37] Adding a 6-component mathematical object of his own invention called a “*Traktor,*” Minkowski went on to extend Maxwell’s theory to cover matter in motion.[38] Letting σ denote a four-current density for matter, and p an antisymmetric second-rank tensor Minkowski called a “*Polarisationstraktor,*” Minkowski’s source equations read:

$$\frac{\partial p_{1j}}{\partial x_1} + \frac{\partial p_{2j}}{\partial x_2} + \frac{\partial p_{3j}}{\partial x_3} + \frac{\partial p_{4j}}{\partial x_4} = \sigma_j - \varrho_j.$$

Ideas for the reformulation of electrodynamics such as these formed the basis of Minkowski’s first publication in theoretical physics, presented six weeks later to the Göttingen Academy of Science, and entitled: “*Die Grundgleichungen für die electromagnetischen Vorgänge in bewegten Körpern*” (hereafter *Die Grundgleichungen*) [54].

A few key ideas that appear in the latter publication were still in embryonic form at the time of his lecture to the mathematical society. For instance, Minkowski was then unable to define either a velocity four-vector or a four-force, an incapacity that implies he did not yet conceive of particle motion in terms of a worldline.[39] Consequently, Minkowski’s four-dimensional program must have been based upon the possibility of expressing the laws of electrodynamics in terms of (what would later be known as) four-vectors and six-vectors, and not on four-dimensional mechanics, since Minkowski’s formulation of the

[36] H. Minkowski, “Funktionentheorie,” Arc. 4°1712, JNUL, noted by Pyenson [71, p. 76].

[37] In fact, Poincaré used his four-space to study only the law of gravitational attraction, neglecting all applications to electrodynamics.

[38] The Traktor’s six components were defined via the 4-vector potential, using a two-index notation: $\psi_{jk} = \partial\psi_k/\partial x_j - \partial\psi_j/\partial x_k$, noting the antisymmetry relation $\psi_{kj} = -\psi_{jk}$, and zeros along the diagonal $\psi_{jj} = 0$. In this way, the Traktor components ψ_{14}, ψ_{24}, ψ_{34}, ψ_{23}, ψ_{31}, ψ_{12} match the field quantities $-iE_x$, $-iE_y$, $-iE_z$, B_x, B_y, B_z. When written out in full, one obtains, for example, $\psi_{23} = \partial\psi_3/\partial x_2 - \partial\psi_2/\partial x_2 = B_x$.

[39] Minkowski originally defined velocity as a four-component entity $(w_x, w_y, w_z, i\sqrt{1-w^2})$, where w is ordinary velocity; for details, see [91].

latter required the notion of worldlines in spacetime, which he did not possess when he first announced this program.

Minkowski corrected his mistaken four-velocity vector sometime between 5 November 1907 and 21 February 1908, when he delivered the manuscript of *Die Grundgleichungen* to the printer. The structure of this paper reflects his discovery process, in that the topic of spacetime mechanics is relegated to an appendix. It is in this appendix that Minkowski laid out his signature "space-time" terminology of spacetime points, spacetime filaments, and spacetime lines (i.e., *Raum-Zeitpunkte, Raum-Zeitfaden, Raum-Zeitlinien*).

With respect to the concept of a spacetime line, Minkowski noted that the direction of such lines is determined at every spacetime point. Here Minkowski introduced the notion of "proper time" (*Eigenzeit*), τ, expressing the increase of coordinate time dt for a point of matter with respect to $d\tau$:

$$d\tau = \sqrt{dt^2 - dx^2 - dy^2 - dz^2} = dt\sqrt{1 - \mathfrak{w}^2} = \frac{dx_4}{w_4},$$

with units chosen such that $c = 1$, and where $\mathfrak{w}^2$ is the square of ordinary velocity, $dx_4 = idt$, and $w_4 = i/\sqrt{1 - \mathfrak{w}^2}$, which corrects, incidentally, the flawed definition of this fourth component of four-velocity given by Minkowski in his November 5 lecture. Proper time was then defined as the integral of $d\tau$, evaluated between two spacetime points on a spacetime line:

$$\int d\tau = \int \sqrt{-(dt^2 - dx^2 - dy^2 - dz^2)}.$$

The formal proximity between Minkowski's definition of proper time, on the one hand, and Einstein's formula for relative time [18, p. 904] on the other hand:

$$\tau = t\sqrt{1 - \left(\frac{v}{V}\right)^2},$$

where $v = \mathfrak{w}$ in Minkowski's notation (and $V = 1$), is quite suggestive. Yet Minkowski preferred to credit Lorentz in this instance, describing proper time as a "generalization of the concept of local time formed by Lorentz for uniform motion" [54, p. 100].

Along with spacetime lines and proper time, in the appendix to *Die Grundgleichungen* Minkowski introduced the lightcone structure of spacetime, the spacetime equations of motion of matter, a principle of least action, and a law of gravitation observationally equivalent to the Newtonian law [91, § 2]. The derivation of the latter law was purely geometric, and thoroughly discursive. While Minkowski's research notes show that he used spacetime diagrams while working out his theory of gravitation, he did not see fit to illustrate his reasoning via figures, and as a result, few scientists were able to follow his arguments.[40]

Mathematical response to *Die Grundgleichungen* was nearly flat at first, as the immense majority of mathematicians had no interest at all in physics. But as far as Hilbert was concerned, Minkowski's work in theoretical physics represented the wave of the future. One month after *Die Grundgleichungen* appeared in the *Göttinger Nachrichten*, Hilbert recommended to Klein that

[40] Math. Archiv 60:6, 139, NSUB.

they bring Minkowski onto the editorial board of *Mathematische Annalen*, arguing that Minkowski's "multifaceted orientation and critical sense" would be of the "greatest value" to the journal [21, p. 135]. When Klein replied that he preferred the Leipzig mathematician Otto Hölder to Minkowski, Hilbert explained further his own preference. Not only did Minkowski know more "up-and-coming young mathematicians" than did Hölder, the board of the *Annalen* needed someone "knowledgeable in modern mathematical physics" [21, p. 137]. Klein may have been swayed by these arguments, since both Hölder and Minkowski soon joined the board of the *Annalen*.

The initial response by physicists to Minkowski's spacetime theory was highly critical, as mentioned in the introduction. The criticism targeted two aspects of Minkowski's work: (1) the novel four-dimensional formalism, which had enabled Minkowski to formulate the first relativistic electrodynamics of moving media, and (2) this new electrodynamics itself, which was thought to be inconsistent with the observational base. Any reviewer of Minkowski's electrodynamics had either to rewrite his equations in a recognizable form, or to provide a precis of his formalism. Einstein and Laub chose the former route, along with Abraham, and Nordström [58]. Abraham rewrote Minkowski's theory in ordinary vector notation for inclusion in the second edition (1908) of his authoritative textbook on the electromagnetic theory of radiation [2], but in a later publication [3], he reproduced a part of Minkowski's formalism. He and Max Born [57] were the only ones to follow this route when discussing Minkowski's electrodynamics of moving media. In fact, Abraham revised essential elements of Minkowski's formalism, as did G.N. Lewis, Sommerfeld, and Laue, while only the Sommerfeld-Laue notation gained a following [91]. With hindsight, it appears that Minkowski committed a tactical error by coupling his formalism to a controversial electrodynamics of moving media.

Minkowski probably came to a similar conclusion himself, as he devoted his very next publication, briefly entitled "*Raum und Zeit,*" almost entirely to elements of his spacetime geometry and mechanics, with merely a nod in the direction of electrodynamics.[41] His lecture introduced a crucial tool for teaching and research in relativity theory: the spacetime diagram (Fig. 2). Diagrams illustrating relative motion via paired triads of rectangular axes were a common sight in contemporary textbooks on mechanics, for example, in Poincaré's *Cinématique* (Fig. 3). Such diagrams did not feature a time axis, since this axis did nothing to illustrate relative motion, time being absolute in Galilean kinematics. Nonetheless, space-time diagrams featuring a temporal axis and one or more spatial axes were familiar from late nineteenth-century chronophotographic motion studies by Étienne-Jules Meray and others.

[41] In "*Raum und Zeit,*" Minkowski provided a new geometric view (via a spacetime diagram) of the Liénard-Wiechert potential in terms of a four-potential, and expressed the four-force between two electrons in arbitrary motion.

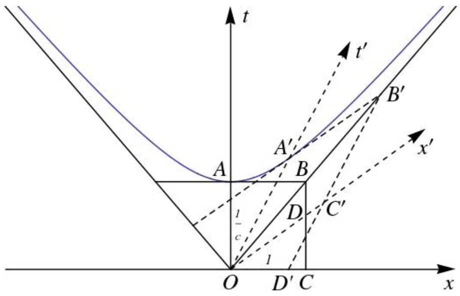

Fig. 2 Fixed and mobile axes in Minkowski spacetime [55]

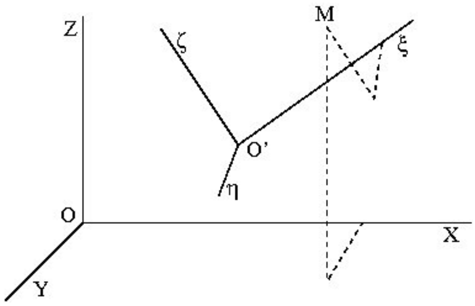

Fig. 3 Fixed and mobile axes in Euclidean space, after Poincaré's *Cinématique* [63]

The fact remains that no graphically-illustrative technique had been proposed for relativity theory, with one exception. On 30 May 1908, Poincaré published a planar projection of what has been dubbed a "light ellipsoid" [70, p. 393]. Although there is no temporal axis in Poincaré's light ellipse, the latter figure bears comparison to a Minkowski spacetime diagram, in that the Lorentz transformation is easily derived from the inscribed geometric relations.[42] Given Minkowski's interest in Poincaré's work in general, and in his theory of relativity in particular, he probably knew of the light ellipse when he wrote "*Raum und Zeit,*" and it is quite plausible that Poincaré's example prompted him to illustrate his own ideas with spacetime diagrams.

The spacetime diagram was a central feature of Minkowski's Cologne lecture, elegantly illustrating several aspects of his theory, including the limit relation between pre-relativist and relativist mechanics. As the value of the constant c approaches infinity, the angle formed by the asymptotes (Fig. 2) becomes more and more oblique, while the the primed space axis x' of the moving frame approaches the x axis of the frame at rest. Then the primed time axis t' of the moving frame may assume any positive direction, replicating (in the limit) the conditions of Newtonian mechanics (Fig. 3). The spacetime diagram was now deployed without restraint; Minkowski even tried to use it to distinguish his understanding of space and time from that of Einstein and Lorentz, with mixed results [89, § 2.4]. In all, the published version of "*Raum und Zeit*" counted four spacetime diagrams, underpinning Minkowski's geometrical argument, and underlining the mathematical nature of his spacetime

[42]Poincaré considered the wave produced by a flash of light from a source in constant rectilinear motion with respect to an observer at rest. At some time t after the flash, a co-moving observer ascertains the radius of the light wave with a measuring rod deformed by Lorentz-FitzGerald contraction. Taking the contraction into account, the co-moving observer concludes that the rectified form of the wave is an ellipsoid [13, p. 38]. When illustrated on a Minkowski map, Poincaré's light ellipsoid corresponds to the projection of the light sphere from a certain constant-time (spacelike) hyperplane of hte co-moving frame to the hyperplane $t = 0$ of a certain frame considered to be at rest.

theory.

4 Concluding remarks

Although Minkowski did not live to see it, the publication of "*Raum und Zeit*" kicked off a wave of publications in relativity theory, the number of papers on relativity having trebled from 1908 (32 papers) to 1910 (95 papers) [89]. This momentous upswing in interest is a complex historical phenomenon, little studied and poorly understood a century after the fact, for which Minkowski's lecture is only one of many sources. While most physicists were shocked by Minkowski's claim that space and time were "doomed to fade away into mere shadows" of the four-dimensional reality of spacetime [55, p. 75], they took notice of his spacetime theory on this occasion. Eventually, physicists – Einstein included – recognized the advantages of Minkowski's geometric approach to relativity, and contributed to a growing corpus of Minkowskian relativity.

Mathematicians, too, contributed to this corpus in ever-greater numbers, much as Minkowski had hoped. Between 1909 and 1915, sixty-five mathematicians wrote 151 articles on relativity theory (excluding the theory of gravitation), or one out of every four articles published in this domain. By 1913, mathematicians publishing articles worldwide on the theory of relativity (22) outnumbered theoretical physicists (16), as well as other physicists (15) [89, § 3.2]. Following Einstein's discovery of the field equations of general relativity in November 1915, further possibilities for mathematical contributions to physics emerged, particularly in differential geometry.

The response to Minkowski's contributions to relativity provides a sharp image of the disciplinary frontiers of mathematics and physics at the turn of the twentieth century. Einstein and Laub's decision to excise Minkowski's four-dimensional formalism from his electrodynamics of moving media, on one hand, and Minkowski's decision to excise electrodynamics from his theory of spacetime, on the other hand, reflect their conceptions of what was essential to physics and mathematics. Likewise, Hilbert and Klein's decision to include Minkowski on the board of *Mathematische Annalen* turned in part on the idea that mathematicians should be encouraged to do mathematical physics.

Effectively displacing the disciplinary frontiers of physics and mathematics, Minkowski's Cologne lecture successfully focused attention on the transformation group leaving invariant the laws of physics, much as Poincaré and Einstein had tried to do three years earlier. Unlike his predecessors, Minkowski described relativity theory as *mathematical* in essence, and he provided a comprehensible graphic illustration of the kinematics of the Lorentz group.

From a modern standpoint, it is quite easy to imagine how some nineteenth-century mathematician might have discovered spacetime. There is a significant precedent for such counterfactual speculation, beginning with the Cologne lecture, where Minkowski first suggested that a mathematician might have come up with the theory of relativity by noticing that natural laws are covariant with respect to the Lorentz group. Of course, more than this was required for the discovery of spacetime, as Minkowski knew quite well from personal experience. His own path to the study of relativity, as we have seen, began in earnest only in 1907, five full years after he had joined the Göttingen faculty. Minkowski's years in Göttingen precipitated the entry of pure mathematicians into the field of theoretical physics, but did not signal any corresponding deriliction of pure mathematics on the part of Hilbert and Minkowski, who continued to cultivate the latter domain with vigor. Theoretical physics was for them not a branch of mathematics, but a source of interesting problems, which they felt they could study with profit for both mathematics and physics.

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