Introduction to Poincaré's Three Supplements

Jeremy J. Gray and Scott A. Walter

Published in *Three Supplements on Fuchsian Functions* by Henri Poincaré, edited by Jeremy J. Gray and Scott A. Walter, Berlin: Akademie-Verlag, 1997, 1–25.

Introduction

The three *suppléments* by Poincaré, written in 1880, are published here for the first time. They document his discovery of automorphic functions and the important role non-Euclidean geometry can play in complex function theory. They precede his published papers of 1881 on the subject, and they show in detail how he made and exploited a succession of insights into what was to become his first major contribution to mathematics.

To assist in the understanding of these papers we first indicate something of Poincaré's life at the time, and describe the context in which he was working. Then we summarize and analyze the mathematical content of the *suppléments*, focusing on what is new and significant in what he did. We indicate also how these discoveries made their way into the many papers that Poincaré was to publish on this subject. Lastly, we indicate briefly how these *suppléments* came to be rediscovered, and conjecture how they were lost.

The context

Poincaré celebrated his twenty-sixth birthday on April 29, 1880. At that time he was *Chargé de cours d'Analyse mathématique* at the Caen Faculty of Science. After graduating second in his class at the *École polytechnique* in 1875 (poor marks in descriptive geometry cost him the top position), Poincaré went on to the *École des mines* in Paris. This was the normal career path for the top graduates of *Polytechnique*; in Poincaré's class only the top three students made it into *Mines* (which must have added spice to the competition for grades). Once Poincaré was enrolled in mining school, his mentor Ossian Bonnet intervened with the school administration on his behalf; he asked that Poincaré be allowed to skip some required courses in docimasy in favor of lectures in mathematics across the street at the university he taught at, the Sorbonne. When the director of *Mines* personally informed Poincaré that the study of mathematics was incompatible with his status as a student engineer, he accepted the decision with magnanimity.

A path leading from the *École polytechnique* and the *École des mines* to a university teaching career had been worn by some of the professors Poincaré most admired, including Camille Jordan and Alfred Cornu. It is unlikely that he ever considered a career as a mine inspector, but that is exactly what he became once he obtained the diploma from *Mines*. Not that this was a shameful occupation. The mine inspector in late nineteenth century France was a highly esteemed individual, one who jeopardized his life in the service of the country. The dangerous nature of this occupation may be judged from the fact that neither of Poincare's two comrades from *Polytechnique* attained the age of thirty.

For all that he impressed everyone who met him with his quickness of mind, Poincaré was not a prodigy. Nor was he particularly well read, preferring to make his own way through contemporary mathematics. By 1880, he still had only two short publications to his name, although in 1878 he had written a doctoral thesis that Darboux, one of his examiners, said contained the material for several good theses (Darboux et al, eds, 1916, xxi). Rather more sharply, Darboux also observed that the methods in the thesis often fell short of rigorous proof, and had urged Poincaré to tighten it up. Instead Poincaré replied that there were other ideas he would rather work on, and in the event, the thesis was not published (until it appeared in the first volume of his *Œuvres*, in 1928). Nominally devoted to extending Kovalevskaya's theorem about partial differential equations in the complex domain, where it foreshadowed part of the analysis of celestial mechanics he later gave in his prize-winning memoir of 1889, the thesis also contained important results on lacunary series and algebroid functions, which came to play an important part in the study of complex functions of several variables. (For a rich account of the writing of this memoir, see Barrow-Green 1997).

The thesis permitted Poincaré to give a course in analysis at the *Faculté des sciences* at Caen; he was officially released from his duties as a mine inspector on December 1, 1879. He was by then thinking about the global theory of real differential equations which he was to develop and incorporate into his celestial mechanics (see Gilain 1977, 1991). But he was also engaged with the theory of differential equations in the complex domain, the subject of his paper of 1878. The theory was then the central topic in the study of ordinary differential equations (see, for example, Gray 2000). The French authorities on the subject had been Briot and Bouquet, but more recently, leadership had passed to a student of Kummer's, much influenced by Weierstrass, the German Lazarus Immanuel Fuchs. Fuchs had succeeded in 1866 in classifying those ordinary linear differential equations whose solutions have fixed singular points at which they have, at worst, finite poles. This is a large class of differential equations which contains the celebrated hypergeometric equation. Fuchs's work on this topic formed the natural generalization of Riemann's paper on the so-called *P*-functions. Since then, Fuchs had solved a number of related problems, including some concerned with elliptic integrals and modular functions by means of his theory. This brought him into contact with Hermite.

Hermite's contact with Fuchs was an important route for German ideas to reach France. He was not comfortable with the methods of Riemann, and barely mentioned them in either his *Cours d'analyse* (1873) or in his later course (1881). But if, unhappily for French mathematics, he shared with Fuchs a failure to understand Riemann's more profound ideas, his appreciation of Fuchs's work was to benefit Poincaré. Hermite was the most influential French mathematician of his generation, alongside Bertrand. Bertrand occupied more prestigious positions, but Hermite's research carried greater weight. Between them they could more-or-less decide who was to get the call to Paris and who was to languish in the provinces. Hermite's failure with Riemann goes some way in explaining why Riemann's ideas had to wait for Picard and Poincaré in the 1880s before they took off in France, well after their adoption by the leading Italian mathematicians.

One way a mathematician like Hermite exerted his influence was through the prize competitions run by the *Académie des sciences* in Paris. It was the custom throughout the nineteenth century for the *Académie* to announce various prizes in mathematics. Typically, a title would be announced, with a panel of judges, and a cut-off date some two years hence. A system of sealed envelopes and mottoes was used to try to ensure anonymity. The entries would be judged, and perhaps a prize would be awarded. But it might well happen that no entry was thought worthy. In that case the essay might be re-announced. On occasion, the prize would go to someone for their work, whether or not it fit the title—this was the case when Abel and Jacobi won the prize in 1830. To avoid this sort of embarrassment, the essays would sometimes be devised with a likely winner in mind, as was the case when Kovalevskaya won the *Prix Bordin* (see Cooke 1984). In 1878, Hermite took the opportunity to set an essay on Fuchs's work that he may well have thought would catch the interest of Poincaré, and of course Poincaré had been Hermite's student at the *École polytechnique*.

A prize competition was thus announced by the Academy in 1878. The question set was "To improve in some important way the theory of linear differential equations in a single independent variable" ("*Perfectionner en quelque point important la théorie des équations différentielles linéaires à une seule variable indépendante*"). The closing date was 1880; and the panel of judges comprised Bertrand, Bonnet, Puiseux and Bouquet, with Hermite as *rapporteur*.

On March 22, 1880, Poincaré submitted a memoir on the real theory, which he withdrew on June 14, before the examiners could report on it. It would seem that his imagination had been captured by the very different complex case, which he wrote up and submitted on May 29, 1880.

This essay was only to be published posthumously, in *Acta Mathematica* 39 (1923) and in the first volume of his *Œuvres* (Appell & Drach, eds, 1928, 336–372), along with his doctoral thesis.

The next day he wrote the first of several letters to Fuchs. Shortly afterwards he had the first breakthrough into the topic of automorphic functions, and wrote the first of the three *suppléments* published in this volume. It is, of course, this connection through Hermite to Fuchs, and Poincaré's patchy reading, that explains why Poincaré chose to call a large class of automorphic functions "Fuchsian". To understand the chain of thought that led to the prize essay and the *suppléments*, it is best to review briefly Fuchs's work and then the original essay.

The work of Fuchs

In a series of papers in 1880 (continuing into 1881, this summary follows Fuchs 1880c, 1904, 191–212; 1880a, 1880b, 1906, 213–218), Fuchs studied the differential equation

$$\frac{d^2y}{dz^2} + P(z)\frac{dy}{dz} + Q(z)y = 0 \tag{1}$$

where P and Q are rational functions of a complex variable z. He took functions $f(z)$ and $\phi(z)$ as a basis of solutions for it, and sought to generalize Jacobi inversion from the context of integrals to differential equations by considering the equations

$$\begin{aligned} \int_{\zeta_1}^{z_1} f(z)dz + \int_{\zeta_2}^{z_2} f(z)dz = u_1 \\ \int_{\zeta_1}^{z_1} \phi(z)dz + \int_{\zeta_2}^{z_2} \phi(z)dz = u_2 \end{aligned} \tag{2}$$

as defining functions of u_1 and u_2:

$$z_1 := F_1(u_1,\ u_2), \quad z_2 := F_2(u_1,\ u_2).$$

By varying the paths of integration he obtained these equations for them:

$$F_i(\alpha_{11}u_1 + \alpha_{12}u_2 + \alpha_1 c,\ \alpha_{21}u_2 + \alpha_{22}u_2 + \alpha_2 c) = F_i(u_1,\ u_2), \quad i = 1, 2$$

where the integers α_{ij} describe the analytic continuation of u_1 and u_2 along paths that cross the cuts joining the singularities of (1) to ∞; α_1 and α_2 are analogous to the periods of an elliptic integral.

Fuchs wished to ensure that the four derivatives $\frac{\partial z_i}{\partial u_j}$ are holomorphic functions of z_1 and z_2 near $z_1 = a$, $z_2 = b$, where a and b are arbitrary distinct points, and that every value $(z_1,\ z_2) \in \mathbb{C}$ can be attained with finite $(u_1,\ u_2) = \mathbb{C}$. For this he said it is necessary and sufficient that at each finite singular point the roots of the associated indicial equation satisfy certain simple conditions (roughly speaking, that they be rational numbers of a precise kind). With increasing obscurity, he then argued that extra conditions on the roots of the indicial equation ensured that the equation

$$\frac{f(z)}{\phi(z)} = \zeta$$

defines z as a single-valued function of ζ and that the equation $f(z_2) - \phi(z_1) - f(z_1)\phi(z_2)$ has only the trivial solution $z_1 = z_2$. In particular, he stipulated that the solutions to the differential equation may not involve logarithmic terms. In an even more special case the number of finite singular points can not be greater than six, and he gave an example where six finite singular points occur. The functions

$$z_1 := F_1(u_1, u_2), \quad z_2 := F_2(u_1, u_2)$$

are then necessarily hyperelliptic, but generally they will not even be Abelian functions, since the differential equation will not be algebraically integrable.

Fuchs's proofs of these assertions proceeded by a case-by-case analysis of each kind of singularity that could occur in terms of the local power series expansions of the functions. As we shall see, Poincaré was to point out that the analysis rapidly becomes confusing and was incomplete, in any case. The condition that no logarithmic terms appear in the solutions to the differential equation even though Fuchs allowed that roots of an indicial equation may differ by 1, an integer, is a strong restriction on the kind of branching that can occur. Fuchs seems to have assumed, or perhaps was only interested in, the case when ζ takes every value in $\mathbb{C}$, not merely in some disc.

As an example of the case when there are six singular points, Fuchs adduced the hyperelliptic integrals

$$y_1 = \int \frac{g(z)}{\sqrt{\phi(z)}} dz, \quad y_2 = \int \frac{h(z)}{\sqrt{\phi(z)}} dz$$

where $\phi(z) = (z - a_1) \dots (z - a_6)$ and ∞ is not a singular point. In this case $g(z)$ and $h(z)$ are linearly independent polynomials of degree 0 or 1 (say $g(z) := 1$, $h(z) := z$). Now $z_1 = F_1(u_1,\ u_2)$, $z_2 = F_2(u_1,\ u_2)$ are hyperelliptic functions of the first kind.

Fuchs was chiefly concerned to study the inversion of equations (2) and was only slightly interested in the function $\zeta = \frac{f(z)}{\phi(z)}$. His obscure papers rather confused the two problems, but they were soon to be disentangled, in the course of a correspondence that the young Poincaré began once he had submitted his essay for the prize competition.

The prize essay

In the essay Poincaré focused on the question of when the quotient $z = \frac{f(x)}{g(x)}$ of two independent solution of a differential equation $\frac{d^2y}{dx^2} = Qy$ defines, by inversion, a meromorphic function x of z. He found Fuchs's conditions were neither necessary nor sufficient, because the nature of the domain of definition of the inverse function had not been adequately considered. It was necessary and sufficient for x to be meromorphic on some domain that the roots of the indicial equation at each singular point, including infinity, differ by an aliquot part of unity (i.e. $\rho_1 - \rho_2 = 1/n$, for some positive integer n). If the domain is to be the whole complex sphere then this condition is still necessary, but it is no longer sufficient. Finding that Fuchs's methods did not enable him to analyze the question very well, as special cases began to proliferate, he sought to give it a more profound study, working upwards from the simplest cases. He began with an example of Fuchs's where the differential equation has two finite singular points and certain exponent differences. These forced x to be a meromorphic single-valued function of z mapping a parallelogram composed of eight equilateral triangles onto the complex sphere, and $z = \infty$ is its only singular point, so x is an elliptic function. The differential equation, Poincaré showed, has in fact an algebraic solution and a non-algebraic solution. This result agrees with Fuchs's theory.

Poincaré next investigated when a doubly-periodic function can give rise to a second-order linear differential equation, and found after a lengthy argument that there was always such an equation having rational coefficients for which the solution was a doubly periodic function having two poles. If furthermore the periods h and K were such that

$$2ß\pi \equiv (\mathrm{mod} h, K)$$

then x would be a monodromic function of z with period $2ß\pi$.

After a further argument Poincaré concluded (79) that there are cases when one solution of the original differential equation is algebraic, and then Fuchs's theory was correct. However, there are also cases when the differential equation has four singular points and elliptic functions are involved; then extra conditions are needed.

However, it might be that the domain of x failed to be the whole z-sphere. Poincaré gave an example to show that this could happen even when the differential equation has only two finite singular points. If the exponent differences are $\frac{1}{4}$

and $\frac{1}{2}$ at the finite points and $\frac{1}{6}$ at ∞, and the finite singular points are joined to ∞ by cuts, then as long as x crosses no cuts z stays within the quadrilateral $\alpha O \alpha' \gamma$ (see Figure 1). The image of the upper and lower half planes are triangles that form a quadrilateral joined along the image of the line joining the singular points.

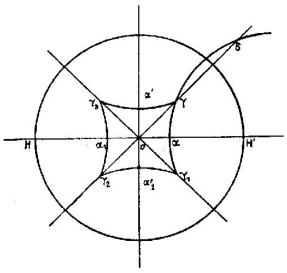

Figure 1

As x is conducted about in its plane, the values of z lie inside the circle HH'. All the images of the upper and lower half planes taken together are quadrilaterals Poincaré described as '*mixtiligne*', with circular-arc sides meeting the circle *HH'* at right angles. For a range of similar differential equations this geometric picture is quite general: curvilinear polygons are obtained with non-re-entrant angles and circular-arc sides orthogonal to the boundary circle. They fill out the domain of the function x in $|z| < OH$, and Poincaré then investigated whether x is meromorphic. This reduces to showing that, as x is continued analytically, the polygons do not overlap. This does not occur if the angles satisfy conditions derived from Fuchs's theory, unless the overlap is in the form of an annular region:

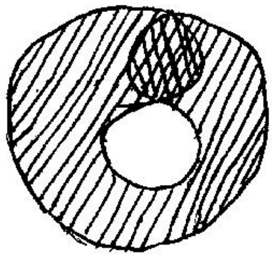

Figure 2

However, if the angles are not re-entrant, this cannot happen, and so x is meromorphic.

The correspondence between Poincaré and Fuchs

The essay out of the way, Poincaré could turn to some of the problems that had occurred to him while reading Fuchs's work. One of his first questions to Fuchs concerned the nature of the inverse function ($z = z(\zeta)$ in Fuchs's notation).[1] Fuchs had claimed that z is always a meromorphic function of $\zeta = \frac{f(z)}{g(z)}$, whether z is an ordinary or a singular point of the differential equation. He showed, in fact, that z is finite at ordinary points and infinite at singular points. Poincaré observed that z is meromorphic at $\zeta = \infty$, which makes $z = z(\zeta)$ meromorphic on the whole ζ-sphere, and so it is a rational function of ζ. This then implies that the original differential equation must have all its solutions algebraic, which Fuchs had expressly denied. It is again a problem of the domain of definition. Poincaré suggested that there were three kinds of ζ-value: those reached by $\frac{f(z)}{g(z)}$ as z traced out a finite contour on the z-sphere; those reached on an infinite contour, and those which are not attained at all. *A priori,* he said, all three situations could occur, and unless the differential equation has only algebraic solutions, the last two would occur. Fuchs's proof only worked for ζ-values of the first kind; however, Poincaré went on, he could show that $z(\zeta)$ was meromorphic even if the other kinds occurred, and he was led to hypothesize: (1) if indeed all ζ-values were of the first kind then z would be a rational function; (2) if there are values of only the first and second kinds, but z is single-valued at the values of the second kind, then Fuchs's theorem is still true; (3) if z is not single-valued or (4) if the values of the third kind occur and so the domain of z is only part of the ζ-sphere, then z is single-valued on D. In this case the ζ-values of the first kind occur inside D. Those of the second kind lie on the boundary of D, and the unattainable values lie outside D. Finally there is a fifth case, when all three kinds of ζ-value occur, but D has this form:

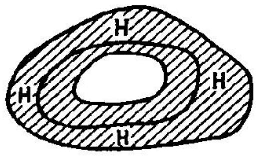

Figure 3

where values of the first kind fill out the annulus. Now, said Poincaré, z will not return to its original value on tracing out a closed curve $HHHH$ in D.

Fuchs replied on the fifth of June. He agreed that his Theorem I was imprecisely worded, and returned to the hypotheses of his earlier *Göttingen Nachrichten* articles about the exponents at the singular points. He added that he excluded paths in which $f(z)$ and $\phi(z)$ both become infinite, which, he said, ensured that the remaining ζ-values filled out a simply-connected region of the ζ-plane with the excluded values on the boundary.

[1] Seven letters in the Poincaré-Fuchs correspondence are published in Julia and Pétiau, eds (1956, 13–25), with an eighth in the photograph on pages 275–276.

Poincaré replied on the twelfth. Finding that some parts of the proof were still obscure he suggested this argument. Let the singular points of the differential equation be joined to ∞ by cuts. The image of this region (when z is not allowed to cross the cuts) is a connected region F_0. If z crosses the cuts no more than m times, then the values of ζ fill out a connected region F_m. As m tends to infinity F_m tends to the region Fuchs called F, and F will be simply-connected if F_m is simply-connected for all m. "Now," asked Poincaré, "is that a consequence of your proof? One needs to add some explanation." He agreed that F_m could not cover itself as it grew in this fashion:

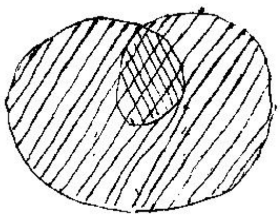

Figure 4

but the proof left open the possibility that the crossing formed an annular region (as in Figure 2, above).

Poincaré said that when there were only two finite singular points it was true that z was a single-valued function, "That I can prove differently," he went on, "but it is not obvious in general. In the case where there are only two finite singular points I have found some remarkable properties of the functions you define, and which I intend to publish. I ask your permission to give them the name of Fuchsian functions." In conclusion, he asked if he might show Fuchs's letter to Hermite.

Fuchs replied on the sixteenth, promising to send him an extract of his forthcoming complete list of the second order differential equations of the kind he was considering. This work, he said, makes any further discussion superfluous. He was very interested in the letters, and very pleased about the name. Of course his replies could be shown to Hermite.

The reply shows once again the important difference of emphasis between the two mathematicians. Fuchs was chiefly interested in studying functions obtained by inverting the integrals of solutions to a differential equation, thus generalizing Jacobi inversion. For him it was only by the way that one might ask that the inverse of the quotient of the solutions be single-valued. This is a requirement that imposes extra conditions. Poincaré was interested in the global nature of the solutions to differential equations, and so it was only the special case that was of interest, and he gradually sought to emancipate it from its Jacobian origins. It is not without irony that we find the young man gently explaining about analytic continuation and the difference between single-valued and unbranched functions, to someone who had consistently studied and applied the technique for fifteen years.

Poincaré's reply of the nineteenth of June clearly demonstrates this difference of emphasis. Taking the condition on the exponents to be what Fuchs had indicated in his letter, Poincaré wrote that he had found that when the differential equation was put in the form $y'' + qy = 0$, then at all the finite singular points the exponent difference was an aliquot part of 1 and not equal to 1, and there were no more than three singular points. If there was only one, z was necessarily a rational function of ζ. If there were two, and the exponent differences were ρ_1, ρ_2, and ρ_3 at infinity, then either $\rho_1 + \rho_2 + \rho_3 > 1$, in which case z is rational in ζ, or $\rho_1 + \rho_2 + \rho_3 = 1$, in which case z was doubly periodic. Even in this case there were difficulties, as he showed with an example. Finally, if there were three finite singular points, then the exponents would have to be -2 and 0, and at infinity they would be $\frac{3}{2}$ and 2. But although these satisfied Fuchs's criteria, z was not a single-valued function of ζ, so the theorem is wrong. Poincaré therefore proposed to drop the requirement that Fuchs's functions $z_1 + z_2$ and $z_1 \cdot z_2$ be single-valued in u_1 and u_2. He went on to say that this gives a "much greater class of equations than you have studied, but to which your conclusions apply. Unhappily, my objection requires a more profound study, in that I can only treat two singular points." Dropping the conditions on the sum and product functions z_1+z_2 and $z_1 \cdot z_2$ admits the possibility that the exponent differences ρ_1, ρ_2, and ρ_3 satisfy $\rho_1+\rho_2+\rho_3 < 1$. Now z is neither rational nor doubly periodic, but is still single-valued. Poincaré explained, "These functions I call Fuchsian, they solve differential equations with two singular points whenever ρ_1, ρ_2, and ρ_3 are commensurable with each other. Fuchsian functions are very like elliptic functions, they are defined in a certain circle and are meromorphic inside it." On the other hand, he concluded, he knew nothing about what happened when there were more than two singular points.

We do not have Fuchs's reply, but Poincaré wrote to him again on the thirtieth of July to thank him for the table of solutions which, he said "lifts my doubts completely." Or perhaps, not quite completely, for he went on to point out a condition on some of the coefficients of the differential equations which Fuchs had not stated explicitly in the formulation of his theorems. As for his own researches on the new functions, he remarked that they "present the greatest analogy with elliptic functions, and can be represented as the quotient of two infinite series in infinitely many ways. Amongst these series are those which are entire series playing the role of Theta functions. These converge in a certain circle and do not exist outside it, as thus does the Fuchsian function itself. Besides these functions there are others which play the same role as the Zeta functions in the theory of elliptic functions, and by means of which I solve linear differential equations of arbitrary orders with rational coefficients whenever there are only two finite singular points and the roots of the three determinantal equations are commensurable. I have also thought of functions which are to Fuchsian functions as Abelian functions are to elliptic functions, and by means of which I hope to solve all linear equations when the roots of the determinantal equations are commensurable. In the end, functions precisely analogous to Fuchsian functions will give me, I think, the solutions to a great number of differential equations with irrational coefficients."

The correspondence winds down at this point, and Poincaré's last letter (March 20, 1881) merely announces that he will soon publish his research on the Fuchsian functions, which partly resemble elliptic functions and partly modular functions, and on the use of Zetafuchsian functions to solve differential equations with algebraic coefficients. In fact, his first two articles on these matters had by then already appeared in the *Comptes rendus de l'Académie des sciences*.

The first supplement

Received by the Academy on the twenty-eighth of June, 1880, the first of Poincaré's three supplements is eighty pages in length. It begins by discussing the validity of Fuchs's theorem when there are only two finite singular points, and all the exponent differences are reciprocals of integers, say ρ_1, ρ_2, and r. Poincaré concentrated on the case when $\rho_1 + \rho_2 + r < 1$, to which he had just been led. In this case y maps the complex x-sphere onto a quadrilateral Q, and under analytic continuation Q can be mapped onto a neighboring copy of itself obtained by rotating it through an angle of $\frac{2\pi}{\rho_1}$ about an appropriate vertex. Another copy is obtained by a rotation through $\frac{2\pi}{r}$ about another vertex. Poincaré called these rotations M and N, and observed that the copies of Q obtained by analytic continuation fill out a disc, and that each copy of Q can be reached by a succession of crab-wise rotations (8):

$$M^{L_1} N^{K_1} M^{L_2} N^{K_2} \dots .$$

All these motions preserve the boundary circle, and taken together they form a group (9).

In this connection, Poincaré remarked (14–15):

> There are close connections with the above considerations and the non-Euclidean geometry of Lobachevsky. In fact, what is a geometry? It is the study of the group of operations formed by the displacements to which one can subject a body without deforming it. In Euclidean geometry the group reduces to the rotations and translations. In the pseudogeometry of Lobachevsky it is more complicated.
>
> Indeed, the group of operations formed by means of M and N is isomorphic to a group contained in the pseudogeometric group. To study the group of operations formed by means of M and N is therefore *to*

> *do the geometry of Lobachevsky*. Pseudogeometry will consequently provide us with a convenient language for expressing what we will have to say about this group.[2] (Emphasis in the original).

Poincaré proceeded to develop the convenient language of non-Euclidean geometry, defining points, lines, angles, and equality of figures — two figures are equal if one is obtained from another by a non-Euclidean transformation. Since the copies of Q do not overlap, the inverse function $x = x(y)$ is a function "which does not exist outside the circle and which is meromorphic inside this circle."[3] Poincaré continued:

> I propose to call this function a Fuchsian function. ... The Fuchsian function is to the geometry of Lobachevsky what the doubly periodic function is to that of Euclid.[4]

Such functions only illuminate the study of differential equations if they can be defined independently of the equations. This Poincaré proceeded to do by means of the Fuchsian series he introduced. He let H be an arbitrary rational function and K be an arbitrary combination of M's and N's. He let z and ζ denote two variable quantities inside the boundary circle, and introduced the sum

$$\sum H(zK) - H(\zeta K)$$

taken over all distinct operations K (which, as he observed, is not the same as taking all combinations of M's and N's). He showed that the series was convergent by an ingenious argument concerning the non-Euclidean area and Euclidean perimeter of the region composed of copies of Q lying within a non-Euclidean circle of increasing radius. Because the perimeter tends to a finite amount the integral

$$\int \left(\frac{f'(t)}{f(t) - f(z)} - \frac{f'(t)}{f(t) - f(\zeta)} \right) \frac{dt}{t - v}$$

taken along it remains finite, and so Poincaré was able to conclude (30):

[2] "Il existe des liens étroits entre les considérations qui précèdent et la géométrie non-euclidienne de Lobatchewski. Qu'est-ce en effet qu'une Géométrie ? C'est l'étude du groupe d'opérations formé par les déplacements que l'on peut faire subir à une figure sans la déformer. Dans la Géométrie euclidienne ce groupe se réduit à des rotations et à des translations. Dans la pseudogéométrie de Lobatchewski il est plus compliqué. Eh bien, le groupe des opérations combinées à l'aide de M et de N est isomorphe à un groupe contenu dans le groupe pseudogéométrique. Étudier le groupe des opérations combinées à l'aide de M et de N, c'est donc *faire de la géométrie de Lobatchewski*. La pseudogéométrie va par conséquent nous fournir un langage commode pour exprimer ce que nous aurons à dire de ce groupe." Note that *isomorphe* here is used in Jordan's sense to mean what would now be called "monomorphic".

[3] "Qui n'existe pas à l'extérieur du cercle ... et qui est méromorphe à l'intérieur de ce cercle."

[4] "Je propose d'appeler cette fonction, fonction fuchsienne. ... La fonction fuchsienne est à la géométrie de Lobatchewski ce que la fonction doublement périodique est à celle d'Euclide."

> ...if $H(z) = \frac{1}{v-z}$ [and] if the order of the terms is suitable, the series we considered at the start is convergent.[5]

This result was not as strong as Poincaré wanted, and in a note between pages 23 and 24 he remarked:

> I have not been able to deduce the results I wanted from the consideration of Fuchsian series; however, I thought I should mention them because I remain convinced that they will find application in the theory of Fuchsian functions ....[6]

However, Poincaré immediately observed (33) that if $f\ (z)$ is a Fuchsian function and y_1 and y_2 are two solutions of the differential equation, then $x = f(z)$, $y_1 = (f'(z))^{\frac{1}{2}}$, $y_2 = (f'(z))^{\frac{1}{2}}$ and $f'(z)$ can only vanish at the singular points of the differential equation.

Then he considered equations where the exponent differences were arbitrary rationals: $2K_1\rho_1$, $2K_2\rho_2$, and $2Kr$, where K_1, K_2 and K are integers (43). He took two solutions of the equation to be $F(x)$ and $\Phi(x)$, and defined $\theta_1(\zeta) = F(f(x))$, $\theta_2(\zeta) = \Phi(f(z))$, where f is the Fuchsian function from the preceding case. He called the functions θ_1 and θ_2 Zetafuchsians, remarking (49):

> We shall call them Zetafuchsian functions because they seem to us to be analogous to the Zeta functions one considers in the theory of doubly periodic functions.[7]

(He was to repeat this point in his main paper on Zetafuchsian functions, written in 1884.) He developed them as power series in z and observed (58) that they could be used to solve differential equations with rational exponent differences and two finite singular points. Then (61) he introduced the Thetafuchsian series defined by the series

$$\sum H(zK)\left(\frac{dzK}{dz}\right)^m$$

summed over K, where H is a rational function and K an operation of the group described above. He proved the series converged when $m > 1$ by a very similar argument to the earlier one, and remarked (64):

[5]"...si $H(z) = \frac{1}{v-z}$, [et] si l'ordre des termes est convenable la série que nous avons considérée au début est convergente."

[6]"Je n'ai pu tirer de la considération des séries Fuchsiennes les résultats que j'en attendais; toutefois j'ai cru devoir en parler parce que je reste persuadé qu'on trouvera à appliquer ces séries dans la théorie des fonctions Fuchsiennes ...."

[7]"Nous les appellerons fonctions zétafuchsiennes parce qu'elles nous semblent présenter quelque analogie avec les fonctions zéta que l'on considère dans la théorie des fonctions doublement périodiques."

> I call this series the Thetafuchsian series because of its numerous analogies with the $theta$ functions.[8]

They were of two kinds, one holomorphic in the circle if H has no poles inside the circle, and the other meromorphic when H does have poles inside the circle. Moreover (66):

> The quotient of two Thetafuchsian series (corresponding to the same value of m) is a rational function of the Fuchsian function.[9]

Then Poincaré defined "*Thétazéta*" series, which are to Zetafuchsians what Theta-fuchsians are to Fuchsian functions. Finally he summarized the work so far, which had taken him a long way towards the creation of classes of analytic functions that solve many kinds of linear differential equation with algebraic coefficients. Poincaré stressed in particular that the new functions allowed one to integrate the hypergeometric equation whenever the exponent differences are rational and no logarithmic term appears in the solution. (The term "hyper-geometric" was never used by Poincaré in 1880).

He also defended the use of non-Euclidean geometry, although he pointed out that one could eliminate it if one wished. This last remark may well have been intended for Joseph Bertrand, who was on the jury, and whose former belief in the possibility of a demonstration of the parallel postulate was common knowledge, thanks to the Carton affair. This amusing episode, recently described by Pont (see Pont 1986, 637–660), began when Jules Carton, a professor of mathematics at St. Omer, sent Bertrand a proof of the parallel postulate, which Bertrand endorsed when he presented it to the *Académie des sciences* during the meeting of December 20, 1869 (Bertrand 1869). He compounded his error by publishing a short note of his own simplifying Carton's proof (Bertrand 1870). Darboux, Hoüel, and Beltrami, who were just then actively involved in bringing non-Euclidean geometry to France, were appalled, and others were drawn in. The affair reached the newspapers, and finally it was demonstrated publicly not only that Carton's supposed proof was not new (it had been published by an Italian mathematician, Minarelli 1849), but that it was, of course, fallacious. Bertrand withdrew his support, but one may suppose that it was prudent of Poincaré not to insist on the importance of non-Euclidean geometry for his new work.

[8] "Cette série, je l'appelle série thétafuchsienne à cause de ses nombreuses analogies avec les fonctions θ."

[9] "Le quotient de deux séries thétafuchsiennes (correspondant à une même valeur de m) est une fonction rationnelle de la fonction fuchsienne."

The second supplement

Twenty-three pages in length, the second supplement made its way to the Academy on the sixth of September, 1880. With disarming honesty, it begins:

> I fear that my first supplement was lacking in clarity, and believe that it is not pointless, before generalizing the results obtained, to go over these same results again in order to provide some additional explanations.[10]

These further elucidations took the form of an explicit description of the non-Euclidean geometry of the disc, defining point, line, angle, distance between two points (the cross-ratio definition of the projective approach) and area (as a double integral). He then observed that the maps preserving these quantities (and the boundary circle) are precisely the maps of the form

$$z' = \frac{\alpha z + \beta}{\gamma z + \delta},$$

and he called them "*mouvements pseudogéométriques*", distinguishing between rotations (which have two real fixed points) and translations (which have none). The choice of the word 'real' (*réel*) was unfortunate; he plainly meant 'point inside or outside the circle' as opposed to points on it, which are at infinity in non-Euclidean geometry.

Then he turned to the differential equations he had studied, and the decomposition of the disc into triangles whose angles are aliquot parts of π. He referred to his two proofs that such a decomposition was possible, the first in the essay itself and the other in the first supplement, as follows:

> The first of these demonstrations would not extend to the more general case that I wish to treat; the second is not rigorous. That is why I think it will be useful to give a third demonstration.[11]

The matter that Poincaré had left obscure consisted of showing that every point inside the fundamental circle does lie in some copy of the quadrilateral Q. He now proved it rigorously by showing explicitly how to cover a path from a given point D to the center O, by a finite number of copies of Q; the finitude derived ultimately from the fact that OD has finite non-Euclidean length (7).

[10] "Je crains d'avoir manqué de clarté dans mon premier supplément et je ne crois pas inutile, avant de généraliser les résultats obtenus, devoir revenir sur ces résultats eux-mêmes afin de donner quelques explications supplémentaires."

[11] "La première de ces démonstrations ne s'étendrait pas au cas plus général que j'ai l'intention de traiter; la seconde n'est pas rigoureuse. C'est pourquoi je crois utile d'en donner encore une troisième démonstration."

The first novelty in the supplement was the decomposition of the disc into polygons with angles aliquot parts of π. As with the case of triangles, it is necessary to show that the region of the polygons does not contain any overlaps. When there are no overlaps, the corresponding function is single-valued and continuous on the boundary and takes the same value at corresponding points. Poincaré's comment at this point is most interesting when one recalls that "*monogène*" means analytic (15-16):

> There is always a function that satisfies the conditions stated above. This would not be obvious if we had required our function Φ to be monogenic, but we did not do this; in fact, although there are monogenic functions satisfying the stated conditions, as it will be seen later, I have not made this hypothesis because I have no use for it, and because I am not yet in a position to prove the existence of such functions.[12]

This reveals one of the more delightful gaps in Poincaré's education, for it shows that he did not then know the Riemann mapping theorem. This result asserts that any simply-connected domain in the complex plane which is not the whole plane is equivalent, from the standpoint of complex function theory, to the interior of the unit disc.

Then, Poincaré abruptly stated the connection with the theory of quadratic forms (17). He supposed T was a matrix ("$substitution$") with integer coefficients which preserved an indefinite ternary quadratic form Φ, and S a linear substitution sending $\xi^2 + \eta^2 - \zeta^2$ to Φ. Then STS^{-1} maps the quadratic form $\xi^2 + \eta^2 - \zeta^2$ to itself. Suppose that it sends (ξ, η, ζ) to take over (ξ', η', ζ'). The quantities

$$z = \tfrac{\xi}{\zeta} + \sqrt{-1}\tfrac{\eta}{\zeta}, \quad z' = \tfrac{\xi'}{\zeta'} + \sqrt{-1}\tfrac{\eta'}{\zeta'}$$

are related by a transformation $z = \zeta K$ of the non-Euclidean plane for which $\xi^2 + \eta^2 - \zeta^2 < 0$. Poincaré remarked (19):

> All the points $z \cdot K$ are the vertices of a polygonal net obtained by decomposing the pseudogeometrical plane into mutually congruent pseudogeometrical polygons. The substitutions K are those that transform the polygons into each other, or even, as we shall see below, those that reproduce the functions that we are going to define.[13]

[12]"Il existe toujours une fonction qui satisfait aux conditions énoncées plus haut. Cela ne serait pas évident si nous avions assujetti la fonction Φ à être monogène, mais nous ne l'avons pas fait; en effet bien qu'il existe des fonctions monogènes satisfaisant aux conditions énoncées, ainsi qu'on le verra plus loin, je n'ai pas fait cette hypothèse parce qu'elle m'est inutile, et parce que je ne serais pas encore en état de démontrer l'existence de semblables fonctions."

[13]"Tous les points $z \cdot K$ sont les sommets d'un réseau polygonal obtenu en décomposant le plan

He gave no proof of these claims, nor indeed that the sheets of the hyperboloid provide a model of non-Euclidean geometry in the z-plane — the proof of the latter fact is quite easy — but proceeded at once to generalize his earlier definition of Thetafuchsian functions. Now a polygonal decomposition $P_0 \dots P_i \dots$ is taken to define a group, by saying the transformation K_i maps P_i onto P_0. If $H(z)$ is a rational function then

$$\theta(z) = \sum_i H(zK_i) \left(\frac{dzK_i}{dz} \right)^m$$

defines the new function, for any integer $m > 1$. Convergence was established as before. Poincaré then defined (20) the corresponding Fuchsian functions, $f(z)$, and showed that they took every value including ∞ equally often in the disc, and connected them to differential equations, for $f(z)$ can "serve to integrate a linear differential equation with algebraic coefficients."[14] To show this, he set

$$x = f(z) \quad y_1 = \sqrt{\tfrac{df}{dz}} \quad y_2 = z\sqrt{\tfrac{df}{dz}},$$

and formed the differential equation

$$\begin{vmatrix} y & y_1 & y_2 \\ \dfrac{dy}{dx} & \dfrac{dy_1}{dx} & \dfrac{dy_2}{dx} \\ \dfrac{d^2y}{dx^2} & \dfrac{d^2y_1}{dx^2} & \dfrac{d^2y_2}{dx^2} \end{vmatrix} = 0$$

It has y_1 and y_2 as solutions, and moreover,

$$y_1 \frac{dy_2}{dx} - y_2 \frac{dy_1}{dx} = 1,$$

and

$$y_1 \frac{d^2y_2}{dx^2} + y_2 \frac{d^2y_1}{dx^2} = 0.$$

Indeed it is

$$\frac{d^2y}{dx^2} + \varphi(x) = 0,$$

where

$$\varphi(x) = \frac{dy_1}{dx}\frac{d^2y_2}{dx^2} - \frac{dy_2}{dx}\frac{d^2y_1}{dx^2}$$

pseudogéométrique en polygones pseudogéométriquement égaux entre eux. Les substitutions K sont celles qui transforment ces polygones les uns dans les autres, ou bien encore comme on le verra plus loin, celles qui reproduisent les fonctions que nous allons définir."

[14]"Servir à intégrer une équation différentielle linéaire à coefficients algébriques" (21).

is algebraic as a function of x. Poincaré proved this by showing it was single-valued, invariant under the transformations $z' = zK$; and took only finitely many z-values for each value of $x = f(z)$. In fact, φ is half the Schwarzian derivative of y with respect to x, which Poincaré seems not to have known. Thus Poincaré could conclude this supplement by saying (23):

> To every decomposition of the pseudogeometrical plane into mutually congruent pseudogeometrical polygons there corresponds a function, analogous to the Fuchsian functions, and which enables us to integrate a second-order linear differential equation with algebraic, but irrational, coefficients.
>
> One sees that there are functions, of which the Fuchsian function is only a particular case, which enable us to integrate linear algebraic differential equations. However, in order to determine whether a given equation is integrable in this way, a long discussion would be required which I do not wish to enter into for the moment, but reserve for later.[15]

The third supplement

A mere twelve pages in length, the third supplement reached the Academy on December 20, 1880. Poincaré dealt here with a class of equations which includes the most famous of all the hypergeometric equations: Legendre's equation for the periods of an elliptic integral as a function of the modulus. For this class of equation the fundamental polygon has one or more vertices on the boundary circle; in Legendre's equation all four vertices are at infinity. When the differential equation has just two finite singular points, Poincaré showed how it can be solved by functions obtained by a limiting argument, assuming the validity of some continuity considerations. He argued (9) that the coefficients of an equation of the form

$$\frac{d^2y}{dx^2} = P_0 y,$$

where P_0 is rational in x and the corresponding quadrilateral is finite, can be varied continuously so that the equation becomes a given one of the same form, and the quadrilateral is continuously deformed into the appropriate infinite quadrilateral.

[15] "A toute décomposition du plan pseudogéométrique en polygones pseudogéométriquement égaux entre eux correspond une fonction analogue aux fonctions fuchsiennes et qui permet d'intégrer une équation linéaire de 2^d ordre à coefficients algébriques, mais irrationnels. On voit qu'il y a des fonctions dont la fonction fuchsienne n'est qu'un cas particulier et qui permettent d'intégrer des équations différentielles linéaires algébriques; mais pour déterminer si une équation donnée est intégrable de la sorte, il faudrait une longue discussion que je me réserve d'entreprendre plus tard, mais dans laquelle je ne veux pas entrer pour le moment."

He had shown in the first (still unpublished) part of the memoir that an equation of the form

$$X_p \frac{d^p y}{dx^p} + X_0 y = 0$$

where the X's are polynomials in x, with highest degree m, can always be reduced to an equation of order m and degree p by means of the substitution $y = \int e^{3x} v dz$ where v is a function of z which satisfies a linear equation of order and degree m. Thus any second order equation with rational coefficients can be reduced to one of the second degree, and so to an equation having only two finite singular points, whence it can be solved. Taken together with the other results in the memoir and the supplements they allowed Poincaré to conclude (12):

> Besides, I do not doubt that the numerous equations considered by M. Fuchs in his Memoir in volume 71 of *Crelle's Journal* ... provide an infinity of transcendants ... and that these new functions enable us to integrate all linear differential equation with algebraic coefficients.[16]

Commentary

The three supplements reveal how the discovery of the connection with non-Euclidean geometry enabled Poincaré to advance so rapidly in his research. The discussion in the essay of triangles inside the disc lacks this idea, and is somewhat inconclusive. But the first supplement marks considerable progress in dealing with the general case where the angles of the triangles are $\frac{\pi}{m}$, $\frac{\pi}{n}$, and $\frac{\pi}{p}$ (and $\frac{\pi}{m} + \frac{\pi}{n} + \frac{\pi}{p} < 1$). This Poincaré achieved in two ways: the idea of considering groups of motions enabled him to organize his ideas and formulate hypotheses; the introduction of metrical concepts allowed him to state and sometimes prove convergence theorems for various new power series that he introduced. Although he was consciously modeling his Fuchsian theory on the theory of elliptic functions, the analogy is a subtle one and had not been noticed before. This may well be due to his novel way of obtaining the series. As is also clear from his published papers, Poincaré obtains Riemann surfaces as quotient spaces of the unit disc, not, as was then the accepted way, as branched coverings of the (Riemann) sphere. So he avoids the complicated question of dissecting a Riemann surface and constructing functions on the dissected surface with assigned jumps across the cuts. However, it should be pointed out that Poincaré does not talk about the quotient

[16]"Je ne doute pas d'ailleurs que les nombreuses équations envisagées par M. Fuchs dans son mémoire inséré au Tome 71 du *Journal de Crelle* ... ne fournissent une infinité de transcendantes ... et que ces fonctions nouvelles ne permettent d'intégrer toutes les équations différentielles linéaires à coefficients algébriques." (The reference should presumably be to Vol. 89 of Crelle's *Journal für die reine und angewandte Mathematik*).

space at all at this stage, and there is no hint of the uniformization of algebraic curves.

The date of the first supplement makes it very clear that the realization Poincaré had on boarding the horse-drawn bus at Coutances (see Poincaré 1908, 43–63) was that the "*mixtiligne*" figures in his first essay were conformal versions of non-Euclidean figures. Perhaps he realized that he had shown in the essay how to transform them into the Beltrami-Klein projective figures. It is striking that this realization had escaped Schwarz and Klein for several years. This raises the question of how Poincaré had come to learn of non-Euclidean geometry.

The simple answer, Felix Klein's Erlangen Program (Klein 1872), is surely mistaken. Klein's Erlangen Program defines a geometry as a group acting on a space, and explains that isomorphic group actions give rise to equivalent geometries. Then it seeks to establish that most well-known geometries are special cases of projective geometry, and in particular that non-Euclidean geometry is a geometry whose space is the set of points inside a conic and whose group is the projective transformations mapping the interior of the conic to itself. In papers published at the time Klein showed in more detail how the projective invariant of cross-ratio (which involves four points) can be made to yield a two-point metrical invariant. In the Erlangen Program, however, the emphasis is strongly projective, and metrical geometry is not much discussed. But in Poincaré's work the emphasis is entirely metrical, and there is no suggestion of a hierarchy of geometries; indeed, Euclidean and non-Euclidean geometries are the only ones invoked. It is true that Poincaré first defines the non-Euclidean metric in the disc in a way that involves cross-ratio, but this arises from the fact that his group elements arose naturally as Möbius transformations. There is none of the richness of context that would indicate a direct influence.

Poincaré does not call his view of geometry the Kleinian one, and he was as scrupulous with attributions as his patchy reading and remarkable imagination would allow. The names he mentions are Beltrami and Hoüel. Moreover, the Erlangen Program was only distributed at Erlangen on the occasion of Klein's appointment as a professor there in 1872, and was not the subject of his inaugural address. It is not cited in the literature of the 1870s, and it is even more unlikely that Poincaré, who was not a voracious reader, would have known of it. It did not become well-known until the early 1890s, when later developments, including Poincaré's own subsequent work and that of Sophus Lie made it seem prescient, and when Klein, as the editor of *Mathematische Annalen*, was able to orchestrate its re-distribution. For all these reasons it is very unlikely that the Erlangen Program is the unacknowledged source of Poincaré's philosophy of geometry.

It is harder to decide if Poincaré had read Klein's essay of 1871, which introduced the non-Euclidean group into the story, but in a projective spirit. In his first letter to Klein, written in 1881, Poincaré wrote: "I know how well you are versed in the knowledge of non-Euclidean geometry, which is the real key to the problem we are dealing with."[17] However, this probably only shows that Poincaré found out about Klein's work when he saw that it was relevant to his own concerns, and in view of the more projective cast of Klein's thought this may well be the case. One should not make too much of Poincaré's cross-ratio definition of non-Euclidean distance. His earliest published papers use a different cross ratio (of z_1, z_2, and their images outside the disc, see Poincaré 1881b, reedited in Darboux et al., eds, 1916, 19–22), and it is probable Poincaré made these observations himself. In any case, Poincaré grasped the new geometry more firmly than Klein ever had.

That leaves us with the question of what, if anything, was the source of Poincaré's views on geometry. One clue is the degree to which group theory enters various contemporary formulations of geometry. In the case of Helmholtz's papers, the answer is not at all. Helmholtz discusses rigid-body motions as the source of our knowledge of geometry, but there is no notice taken of the fact that the motions of bodies may be thought of as the action of a group. The same is true of Beltrami's almost-Euclidean talk of superposition. In Klein's case, the concepts of subgroup and isomorphism are brought in to the story. To go to the other extreme, in Lie's case, there is a much more profound analysis, yielding a classification theorem for at least the low-dimensional geometries.

So it would be in the spirit of the Erlangen Program to describe a group action, indicate the appropriate invariants, and establish an isomorphism. It is not in the spirit to fail to mention groups altogether. It goes beyond the spirit to investigate a group in any detail, and well beyond it to seek to analyze all of them. So when in 1880, in the still-unpublished *suppléments* to his essay on linear differential equations, Poincaré simply says that a geometry is a group of operations formed by the displacements of a body that do not deform it, we can see various influences at work. The motion of rigid bodies is an idea vividly presented by both Helmholtz and Beltrami. Even Hoüel in his book on Euclidean geometry wrote in those terms. The conception is more metrical, and narrower than that of Klein.

The sources available to Poincaré included not only work by Hoüel (a friend of Darboux) on Euclidean geometry (Hoüel 1863), but his translations of Beltrami's *Saggio* (Beltrami 1869) and Lobachevsky's *Geometrische Untersuchungen* (Lobachevsky 1866). It is not certain that the work of Helmholtz was known to him, nor is it clear that it would have added anything to what was readily available. With or without Helmholtz's papers, Poincaré could have known from his teachers that geometry is the study of figures in a space that can be moved around rigidly, so that exact superposition is possible and there is a notion of congruence.

[17] "Je sais combien vous êtes versé dans la connaissance de la géométrie non-Euclidienne qui est la clef véritable du problème qui nous occupe."

This idea, which is easier to think through in the metrical than the projective case, works for both Euclidean and non-Euclidean geometry. To anyone aware that thinking group-theoretically is advantageous, it was then natural to observe that the rigid-body motions form a group. This idea could have been had by Jordan, Darboux, Hermite, or Poincaré himself; it could even have been a common-place among the better French mathematicians of the 1870s. There is no need to attribute it to the influence of Klein.

Of these other influences, Beltrami's essay is thoroughly differential-geometric in spirit. It starts from the first fundamental form for a surface of constant negative curvature, and derives formulae for arc length and area on a surface which is represented by the interior of a unit Euclidean disc. In this representation geodesics appear straight (which is why it is sometimes called the Beltrami-Klein projective model, after Klein's re-interpretation of it in 1871), but Beltrami regarded figures as only approximately accurate. He showed that the intrinsic trigonometry of such a surface was that described earlier by Minding and Codazzi, and so the surface carries the non-Euclidean geometry of Lobachevsky. Because Beltrami's presentation is differential-geometric, uses a circular disc, and refers to Lobachevsky but not J. Bolyai or Riemann, it is very likely that this is Poincaré's source. Moreover, Beltrami based the idea of geometry on the exact superposability of figures, which Poincaré also endorsed.

It is clear that geometrical insight always guided his research. First Poincaré dealt with the case where the triangles had angles that were aliquot parts of π, then arbitrary rational parts of π, then, in the final supplement, zero angles. It was more than a convenient language, it underlies the whole appeal to the limiting argument of the third supplement, which is scarcely comprehensible otherwise. It also made possible the connection with the arithmetic of quadratic forms. In this case, as is also clear from the paper he presented to the *Association française pour l'avancement des sciences* in Algiers (Poincaré 1881a, reedited in Châtelet, ed, 1950, 267–274), it is a different model of non-Euclidean geometry, one based on the hyperboloid of two sheets. This model is commonly attributed to Weierstrass and Killing, who knew of it in 1872; Poincaré seems to have come to it independently. The second *supplément* enables us to date his realization to the summer of 1880, probably late August or early September, judging by its abrupt appearance towards the end of the piece.

The fact that the new functions could be used to solve differential equations with algebraic coefficients, together with the flexibility of the continuity method, suggest that the new functions are really functions on a Riemann surface and that almost all Riemann surfaces might be obtainable as quotients of the unit disc. Poincaré did not observe this in the supplements, but in two early papers (April 4 and May 30, 1881; Poincaré 1881d, in Châtelet, ed., 1950, 8–10 and Poincaré 1881c, reedited in Darboux et al., eds, 1916, 16–18) he said that any two Fuchsian functions corresponding to the same group are algebraically related and that he did not know if an arbitrary algebraic curve could be parameterized by Fuchsian functions. Thus we see that Poincaré's use of infinite polygons to prove the uniformization theorem derives from his interest in differential equations, whereas Klein, who was not interested in differential equations, always preferred finite polygons (cf. Freudenthal 1955, 213; Scholz 1980).

The supplements also make apparent astonishing gaps in Poincaré's education, many of which had to be filled by Klein. He clearly did not know Schwarz's work on the hypergeometric equation (Schwarz 1873, 1890, 211–259), in which the first tessellation of the disc by polygons appears. After Poincaré's work, this tessellation can be seen as a non-Euclidean configuration, but Schwarz had missed making this observation. In June, 1881, Klein began a prolonged correspondence with Poincaré, and a running theme of these letters is the choice of names. Klein was adamant that the appellation Fuchsian was undeserved, and in the sixth letter (June 27, 1881, see Julia and Pétiau, eds, 1956, 36) Poincaré admitted that had he known of Schwarz's work, he would have given his new functions a different name, but, as he had already said to Klein, his regard for Fuchs would not now let him change the name. He then went ahead and the same day named a new class of functions 'Kleinian' in the *Comptes rendus* (Poincaré 1881b, reedited in Darboux et al., eds, 1916, 19–22). Klein persisted in his protests against both names, until in letter nineteen (April 4, 1882, Julia and Pétiau, eds, 1956, 55) Poincaré decided he had had enough and protested with a citation from Faust, "*Name ist Schall und Rauch*."

It is also clear that Poincaré had never heard of the Riemann mapping principle, which may indirectly be the negative influence of Hermite. He seems to have suspected such a result ought to be true, but the quotation above makes it clear he could not then prove it. On the other hand he was clearly happy with the idea of automorphic functions, those for which $f\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right) = f(z)$, and their fundamental domains. There is a possible source for this: Dedekind's important paper of 1877 on modular functions. The latter paper virtually emancipated modular functions from the theory of elliptic functions, and since this was a theme dear to Hermite's heart, Poincaré may well have learned about it at *Polytechnique*. If so, then, like Klein, he could easily have added ideas of a group-theoretic kind to it. In any event, he recalled Hermite's work on the modular function, which showed that it is automorphic. The third supplement makes it clear that it was the desire to include this famous function satisfying Legendre's hypergeometric equation that led Poincaré to contemplate his continuity method.

The outcome of the prize competition

The jury, faced with this rush of activity from Poincaré and a more sober memoir from Halphen on differential invariants, along with a number of other essays, opted for sobriety. In awarding Poincaré's essay the second prize, Hermite reported: "... [T]he author successively treated two entirely different questions, of which he made a profound study with a talent by which the commission was greatly struck. The second ... concerns the beautiful and important researches of M. Fuchs ... The results ... presented some lacunæ in certain cases that the author has recognized and drawn attention to in thus completing an extremely interesting analytic theory. This theory has suggested to him the origin of transcendents, including in particular elliptic functions, and has permitted him to obtain the solutions to linear equations of the second order in some very general cases. This is a fertile path that the author has not traversed in its entirety, but which manifests an inventive and profound spirit. The commission can only urge him to follow up his research, in drawing to the attention of the Academy the excellent talent of which they give proof" (see Darboux et al., eds, 1916, 73).

A note on the text of the supplements

Jeremy Gray found the original manuscripts in December, 1979, when he was finishing his doctoral thesis at the University of Warwick. They were in the *Dossier Henri Poincaré* at the *Académie des sciences* in Paris. (JJG adds: I confess that I was completely surprised; it later turned out that I had missed the announcement in the relevant volumes of the *Comptes rendus de l'Académie*, where receipt of each *supplément* was recorded). He communicated his findings to Professor Jean Dieudonné, who very graciously had copies made which he then sent back to Gray. This copy, and Dieudonné's own form the basis of the essays by Gray (1982) and Dieudonné (1982). The account here draws on Gray (1982, 2000), to which the reader is referred for more details.

Poincaré's original essays are hand-written, of course, but the Academy also possesses a fair typewritten version of the first supplement. Professor Dieudonné conjectured that these transcripts might have been made when the original essay was prepared for publication in the first volume of the *Œuvres de Poincaré*, and then forgotten. Be that as it may, the memory of their existence was lost, although they were as secure as the purloined letter, and they even escaped notice during the events of the Poincaré Centenary in 1955.

Editorial policy

Our main concern in editing Poincaré's manuscripts was to produce a legible printed copy, accurately reflecting the original text. A handful of spelling errors have been silently corrected, mostly concerning slips in adjectival accords. The capitalization of "*fonctions fuchsiennes*" has been standardized, in occasional contradiction of the manuscript, which treats this in a haphazard fashion. The paragraph structure of our version reflects our sense for the thematic progression of the text, rather than strong, consistent, objective signal in the manuscripts. Poincaré's own corrections have been flagged with footnote calls. All notation reflects that employed by Poincaré, and the original pagination is shown in brackets. Thus in our version of the first supplement, it is clear that the original pagination is neither continuous nor sequential. There are 79 (non-sequentially numbered) pages, including two page 48's, but neither a page 41 nor a page 42.

Acknowledgment

The three manuscripts submitted by Poincaré for the *Grand Prix des sciences mathématiques* are preserved in the Archives of the Academy of Science in Paris, which generously assented to their publication. The Archives–Centre d'Etudes et de Recherche Henri-Poincaré in Nancy and L. Rollet, in particular, graciously provided assistance to the editors.

References

Académie des sciences de Paris, ed. *Le livre du centenaire de la naissance de Henri Poincaré 1854–1954*. Paris: Gauthier-Villars, 1955.

Appell, P. and Drach, J., eds. *Œuvres d'Henri Poincaré, Volume 1*. Paris: Gauthier-Villars, 1928. URL `http://www.archive.org/details/uvresdehenripoin01poin`.

Barrow-Green, J. E. *Poincaré and the Three Body Problem*. Providence: AMS/LMS, 1997.

Beltrami, E. Essai d'interprétation de la géométrie non-euclidienne. *Annales scientifiques de l'École normale supérieure* 6 (1869): 251–288.

Bertrand, J. Sur la somme des angles d'un triangle. *Comptes rendus hebdomadaires de l'Académie des sciences de Paris* 69 (1869): 1265–1269.

—. Sur la démonstration relative à la somme des angles d'un triangle. *Comptes rendus hebdomadaires de l'Académie des sciences de Paris* 70 (1870): 17–20.

Châtelet, A., ed. *Œuvres d'Henri Poincaré, Volume 5*. Paris: Gauthier-Villars, 1950. URL http://www.archive.org/details/uvresdehenripoin05poin.

Cooke, R. *The Mathematics of Sonya Kovalevskaya*. New York: Springer, 1984.

Darboux, G., Nörlund, N. E., and Lebon, E., eds. *Œuvres d'Henri Poincaré, Volume 2*. Paris: Gauthier-Villars, 1916. URL http://www.archive.org/details/uvresdehenripoin02poin.

Dieudonné, J. La découverte des fonctions fuchsiennes. In *Actes du VIe congrès du regroupement des mathématiciens d'expression latine, Luxembourg, 7–12 septembre 1981*. Edited by Congrès du Groupement des mathématiciens d'expression latine, 3–23. Paris: Gauthier-Villars, 1982.

Freudenthal, H. Poincaré et les fonctions automorphes. In Académie des sciences de Paris (1955), 212–219, 1955.

Fuchs, L. Sur une classe de fonctions de plusieurs variables tirées de l'inversion des intégrales de solutions des équations différentielles linéaires dont les coefficients sont des fonctions rationnelles; Extrait d'une lettre adressée à M. Hermite. *Comptes rendus hebdomadaires de l'Académie des sciences de Paris* 90 (1880a): 678–680. URL http://gallica.bnf.fr/ark:/12148/bpt6k3047v.image.langFR.

—. Sur une classe de fonctions de plusieurs variables tirées de l'inversion des intégrales de solutions des équations différentielles linéaires dont les coefficients sont des fonctions rationnelles; Extrait d'une lettre adressée à M. Hermite. *Comptes rendus hebdomadaires de l'Académie des sciences de Paris* 90 (1880b): 735–736. URL http://gallica.bnf.fr/ark:/12148/bpt6k3047v.image.langFR.

—. Über eine Classe von Functionen mehrerer Variabeln, welche durch Umkehrung der Intergrale von Lösungen der linearen Differentialgleichungen mit rationalen Koeffizienten entstehen. *Journal für die reine und angewandte Mathematik* 89 (1880c): 151–169.

Fuchs, R. and Schlesinger, L., eds. *Gesammelte mathematische Werke von L. Fuchs, Volume 2*. Berlin: Mayer & Müller, 1906. URL http://archive.org/details/gesammeltemathe02fuchgoog.

Gilain, C. *La théorie géométrique des équations différentielles de Poincaré*. Ph.D. dissertation, Université Paris 1, 1977.

—. La théorie qualitative de Poincaré et le problème de l'intégration des équations différentielles. *Cahiers d'histoire et de philosophie des sciences* 34 (1991): 215–242.

Gray, J. The three supplements to Poincaré's prize essay of 1880 on Fuchsian functions and differential equations. *Archives internationales d'histoire des sciences* 32 (1982): 221–235.

—. *Linear Differential Equations and Group Theory from Riemann to Poincaré*. Boston: Birkhäuser, 2d edition, 2000.

Hermite, C. *Cours d'analyse de l'École polytechnique*. Paris: Gauthier-Villars, 1873.

Hoüel, J. Essai d'une exposition rationelle des principes fondamentaux de la géométrie élémentaire. *Archiv der Mathematik und Physik* 40 (1863): 171–211.

Julia, G. and Petiau, G., eds. *Œuvres d'Henri Poincaré, Volume 11*. Paris: Gauthier-Villars, 1956. URL http://www.archive.org/details/ceuvresdeehenrip027739mbp.

Klein, F. *Vergleichende Betrachtungen über neuere geometrische Forschungen*. Erlangen: A. Deichert, 1872. URL http://www.archive.org/stream/vergleichendebe01kleigoog#page/n0/mod

Lobachevsky, N. I. Études géométriques sur la théorie des parallèles. *Mémoires de la Société des sciences physiques et naturelles de Bordeaux* 4 (1866): 8–120.

Minarelli, C. Théorie des parallèles. *Nouvelles annales de mathématiques* 8 (1849): 312–314.

Poincaré, H. Sur les applications de la géométrie non-euclidienne à la théorie des formes quadratiques. *Association française pour l'avancement des sciences* 10 (1881a): 132–138. URL http://www.univ-nancy2.fr/poincare/bhp/pdf/hp1881af.pdf.

—. Sur les fonctions fuchsiennes. *Comptes rendus hebdomadaires de l'Académie des sciences de Paris* 92 (1881b): 1484–1487. URL http://www.univ-nancy2.fr/poincare/bhp/pdf/hp1881crk.pdf.

—. Sur les fonctions fuchsiennes. *Comptes rendus hebdomadaires de l'Académie des sciences de Paris* 92 (1881c): 1274–1276. URL http://www.univ-nancy2.fr/poincare/bhp/pdf/hp1881cri.pdf.

—. Sur une nouvelle application et quelques applications importantes des fonctions fuchsiennes. *Comptes rendus hebdomadaires de l'Académie des sciences de Paris* 92 (1881d): 859–861. URL http://www.univ-nancy2.fr/poincare/bhp/pdf/hp1881crd.pdf.

—. *Science et méthode*. Paris: Flammarion, 1908. URL http://ia700403.us.archive.org/0/items/scienceetmthod00poin/science

—. Extrait d'un mémoire inédit de Henri Poincaré sur les fonctions fuchsiennes. *Acta mathematica* 39 (1923): 58–93. URL http://www.univ-nancy2.fr/poincare/bhp/pdf/hp1923am.pdf.

Pont, J.-C. *L'aventure des parallèles : histoire de la géométrie non euclidienne; précurseurs et attardés*. Bern: Peter Lang, 1986.

Schlesinger, L., ed. *Gesammelte mathematische Werke von L. Fuchs, Volume 1*. Berlin: Mayer & Müller, 1904. URL http://archive.org/details/gesammeltemathe00fuchgoog.

Scholz, E. *Geschichte des Mannigfaltigkeitsbegriffs von Riemann bis Poincaré*. Basel: Birkhäuser, 1980.

Schwarz, H. A. Über diejenigen Fälle, in welchen die Gaussiche hypergeometrische Reihe eine algebraische Function ihres vierten Elements darstellt. *Journal für die reine und angewandte Mathematik* 75 (1873): 292–335.

—. *Gesammelte mathematische Abhandlungen von H. A. Schwarz, Volume 2*. Berlin: Springer, 1890. URL http://ia600306.us.archive.org/30/items/gesammeltemathem02schwuoft/

Time-stamp: "31.07.2014 23:26"