Breaking in the 4-vectors: the four-dimensional movement in gravitation, 1905–1910

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Abstract

The law of gravitational attraction is a window on three formal approaches to laws of nature based on Lorentz-invariance: Poincaré's four-dimensional vector space (1906), Minkowski's matrix calculus and spacetime geometry (1908), and Sommerfeld's 4-vector algebra (1910). In virtue of a common appeal to 4-vectors for the characterization of gravitational attraction, these three contributions track the emergence and early development of four-dimensional physics.

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Introduction

In July, 1905, Henri Poincaré (1854–1912) proposed two laws of gravitational attraction compatible with the principle of relativity and all astronomical observations explained by Newton's law. Two years later, in the fall of 1907, Albert

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Einstein (1879–1955) began to investigate the consequences of the principle of equivalence for the behavior of light rays in a gravitational field (Einstein 1907). The following year, Hermann Minkowski (1864–1909), Einstein's former mathematics instructor, borrowed Poincaré's notion of a four-dimensional vector space for his new matrix calculus, in which he expressed a novel theory of the electrodynamics of moving media, a spacetime mechanics, and two laws of gravitational attraction. Following another two-year hiatus, Arnold Sommerfeld (1868–1951) characterized the relationship between the laws proposed by Poincaré and Minkowski, calling for this purpose both on spacetime diagrams and a new 4-vector formalism.

Of these four efforts to capture gravitation in a relativistic framework, Einstein's has attracted the lion's share of attention, and understandably so in hindsight, but at the expense of a full understanding of what is arguably the most significant innovation in contemporary mathematical physics: the four-dimensional approach to laws of physics. In virtue of the common appeal made by Poincaré, Minkowski, and Sommerfeld to four-dimensional vectors in their studies of gravitational attraction, their respective contributions track the evolving form of four-dimensional physics in the early days of relativity theory.[1] The objective of this paper is to describe in terms of theorists' intentions and peer readings the emergence of a four-dimensional language for physics, as applied to the geometric and symbolic expression of gravitational action.

The subject of gravitational action at the turn of the twentieth century is well-suited for an investigation of this sort. This is not to say that the reform of Newton's law was a burning issue for theorists. While several theories of gravitation claimed corroboration on a par with that of classical Newtonian theory, contemporary theoretical interest in gravitation as a research topic–including the Lorentz-invariant variety–was sharply curtailed by the absence of fresh empirical challenges to the inverse-square law. Rather, in virtue of the stability of the empirical knowledge base, and two centuries of research in celestial mechanics, the physics of gravitation was a well-worked, stable terrain, familiar to physicists, mathematicians and astronomers alike.[2]

The leading theory of gravitation in 1905 was the one discovered by Isaac Newton over two centuries earlier, based on instantaneous action at a distance. When Poincaré sought to bring gravitational attraction within the purview of the principle of relativity, he saw it had to propagate with a velocity no greater than that of light in empty space, such that a reformulation of Newton's law as a retarded action afforded a simple solution.

Newton's law was the principal model for Poincaré, but it was not the only one. With the success of Maxwell's theory in explaining electromagnetic phenomena (including the behavior of light) during the latter third of the nineteenth century, theories of contiguous action gained greater favor with physicists. In 1892, the

Dutch theorist H.-A. Lorentz produced a theory of mobile charged particles interacting in an immobile ether, that was an habile synthesis of Maxwell's field theory and Wilhelm Weber's particle theory of electrodynamics. After the discovery of the electron in 1897, and Lorentz's elegant explanation of the Zeeman effect, certain charged microscopic particles were understood to be electrons, and electrons the building-blocks of matter.[3]

In this new theoretical context of ether and electrons, Lorentz derived the force on an electron moving in microscopic versions of Maxwell's electric and magnetic fields. To determine the electromagnetic field of an electron in motion, Alfred Liénard and Emil Wiechert derived a formula for a potential propagating with finite velocity. In virtue of these two laws, both of which fell out of a Lagrangian from Karl Schwarzschild, the theory of electrons provided a means of calculating the force on a charged particle in motion due to the fields of a second charged particle in motion.[4]

An electron-based analogy to gravitational attraction of neutral mass points was then close at hand. Lorentz's electron theory was held in high esteem by early twentieth-century theorists, including both Poincaré and Minkowski, who naturally catered to the most promising research program of the moment. They each proposed two force laws: one based on retarded action at a distance, the other appealing directly to contiguous action propagated in a medium. All four particle laws were taken up in turn by Sommerfeld.[5]

Several other writers have discussed Poincaré's and Minkowski's work on gravitation. Of the first four substantial synoptic reviews of the two theories, none employed the notation of the original works, although this fact itself reflects the rapid evolution of formal approaches in physics. Early comparisons were carried out with either Sommerfeld's 4-vector formalism (Sommerfeld 1910b; Kretschmann 1914), a relative coordinate notation (De Sitter 1911), or a mix of ordinary vector algebra and tensor calculus (Kottler 1922). No further comparison studies were published after 1922, excepting one summary (North 1965, 49–50), although since the 1960s, the work of Poincaré and Minkowski has continued to incite historical interest.[6] Sommerfeld's contribution, while it inflected theoretical practice in general, and contemporary reception of Lorentz-covariant gravitation theory in particular, has been neglected by historians.

The present study has three sections, beginning with Poincaré's contribution, moving on in the second section to Minkowski's initial response to Poincaré's theory, and a review of his formalism and laws of gravitation. A third section is taken up by Sommerfeld's interpretation of the laws proposed by Poincaré and Minkowski. The period of study is thus bracketed on one end by the discovery of special relativity in 1905, and on the other end by Sommerfeld's paper. While the latter work did not spell the end of either 4-vector formalisms or Lorentz-covariant theories of gravitation, it was the first four-dimensional vector algebra, and represents a point of closure for a study of the emergence of a conceptual framework for four-dimensional physics.

1 Henri Poincaré's Lorentz-invariant laws of gravitation

Poincaré's memoir on the dynamics of the electron (1906), like Einstein's relativity paper (Einstein 1905), contains the fundamental insight of the physical significance of the group of Lorentz transformations, not only for electrodynamics, but for all natural phenomena. The law of gravitation, to no lesser extent than the laws of electrodynamics, fell presumably within the purview of Einstein's theory, but this is not a point that Einstein, then working full time as a patent examiner in Bern, chose to elaborate upon immediately. Poincaré, on the other hand, as Professor of Mathematical Astronomy and Celestial Mechanics at the Sorbonne, could hardly finesse the question of gravitation. In particular, his address to the scientific congress at the St. Louis World's Fair, on 24 September, 1904, had pinpointed Laplace's calculation of the propagation velocity of gravitation as a potential spoiler for the principle of relativity.[7]

There may have been another reason for Poincaré to investigate a relativistic theory of gravitation. In the course of his study of Lorentz's contractile electron, Poincaré noted that the required relations between electromagnetic energy and momentum were not satisfied in general. Raised earlier by Max Abraham, the problem was considered by Lorentz to be a fundamental one for his electron theory.[8]

Solving the stability problem of Lorentz's contractile electron was a trivial matter for Poincaré, as it meant transposing to electron theory a special solution to a general problem he had treated earlier at some length: to find the equilibrium form of a rotating fluid mass.[9] He postulated a non-electromagnetic, Lorentz-invariant "supplementary" potential that exerts a binding (negative) pressure inside the electron, and reduces the total energy of the electron in an amount proportional to the volume decrease resulting from Lorentz contraction. When combined with the electromagnetic field Lagrangian, this binding potential yields a total Lagrangian invariant with respect to the Lorentz group, as Poincaré required.

In accordance with the electromagnetic world-picture and the results of Kaufmann's experiments, Poincaré supposed the inertia of matter to be exclusively of electromagnetic origin, and he set out, as he wrote in §6 of his paper,

> to determine the total energy due to electron motion, the corresponding action, and the quantity of electromagnetic momentum, in order to calculate the electromagnetic masses of the electron.

Non-electromagnetic mass does not figure in this analysis, and consequently, one would not expect the non-electromagnetic binding potential to contribute to the tensorial electromagnetic mass of the electron, although Poincaré did not state this in so many words. Instead, immediately after obtaining an expression for the binding potential, he derived the small-velocity, "experimental" mass from the electromagnetic field Lagrangian alone, neglecting a contribution from the binding potential. The mass of the slowly-moving Lorentz electron was then equal to the electrostatic mass, just as one would want for an electromagnetic foundation of mechanics. This fortuitous result, which revised Lorentz's electron mass value downward by a quarter, was obtained independently by Einstein, using a method that did not constrain electron structure.[10] Although the question of electron mass was far from resolved, Poincaré had shown that the stability problem represented no fundamental obstacle to the pursuit of a new mechanics based on the concept of a contractile electron.

With this obstacle out of the way, Poincaré proceeded as if the laws of mechanics were applicable to the experimental mass of the electron.[11] Noting that the negative pressure deriving from his binding potential is proportional to the fourth power of mass, and furthermore, that Newtonian attraction is itself proportional to mass, Poincaré conjectured that

> there is some relation between the cause giving rise to gravitation and that giving rise to the supplementary potential.

On the basis of a formal relation between experimental mass and the binding potential, in other words, Poincaré predicted the unification of his negative internal electron pressure with the gravitational force, in a future theory encompassing all three forces.[12]

On this hopeful note, Poincaré began his memoir's ninth and final section, entitled "Hypotheses concerning gravitation." Lorentz's theory, Poincaré explained, promised to account for the observed relativity of motion:

> In this way Lorentz's theory would fully explain the impossibility of detecting absolute motion, if all forces were of electromagnetic origin.[13]

The hypothesis of an electromagnetic origin of gravitational force had been advanced by Lorentz at the turn of the century. On the assumption that the force between "ions" (later "electrons") of unlike sign was of greater magnitude at a given separation than that between ions of like sign (following Mossotti's conjecture), Lorentz represented gravitational attraction as a field-theoretical phenomenon analogous to electromagnetism, reducing to the Newtonian law for bodies at rest with respect to the ether. Lorentz's theory tacitly assumed negative energy density for the "gravitational" field, and a gravitational ether of huge intrinsic positive energy density, two well-known sticking-points for Maxwell. Another difficulty stemmed from the dependence of gravitational force on absolute velocities.[14]

Neither Lorentz's gravitation theory nor Maxwell's sticking-points were mentioned by Poincaré in the ninth section of his memoir. Instead, he recalled a well-known empirical fact: two bodies that generate identical electromagnetic fields need not exert the same attraction on electrically neutral masses. Although Lorentz's theory clearly accounts for this fact, Poincaré concluded that the gravitational field was distinct from the electromagnetic field. What this tells us is that Poincaré's attention was not focused on Lorentz's theory of gravitation.[15]

To Poincaré's way of thinking, it was the impossibility of an electromagnetic reduction of gravitation that had driven Lorentz to suppose that all forces transform like electromagnetic ones:

> The gravitational field is therefore distinct from the electromagnetic field. Lorentz was obliged thereby to extend his hypothesis with the assumption that *forces of any origin whatsoever, and gravitation in particular, are affected by a translation* (or, if one prefers, by the Lorentz transformation) *in the same manner as electromagnetic forces*.[16]

It was the cogency of the latter hypothesis that Poincaré set out to examine in detail, with respect to gravitational attraction. The situation was analogous to the one Poincaré had encountered in the case of electron energy and momentum mentioned above, where he had considered constraining internal forces of the electron to be Lorentz-invariant. Such a constraint solved the problem immediately, but Poincaré recognized that it was inadmissible nonetheless, because it violated Maxwell's theory (p. 136). A similar violation in the realm of mechanics could not be ruled out in the case of gravitation, such that a careful analysis of the admissibility of the formal requirement of Lorentz-invariance was called for.

Poincaré set out to determine a general expression for the law of gravitation in accordance with the principle of relativity. A relativistic law of gravitation, he reasoned, must obey two constraints distinguishing it from the Newtonian law. First of all, the new force law could no longer depend solely on the masses of the two gravitating bodies and the distance between them. The force had to depend on their velocities, as well. Furthermore, gravitational action could no longer be considered instantaneous, but had to propagate with some finite velocity, so that the force acting on the passive mass would depend on the position and velocity of the active mass at some earlier instant in time. A gravitational propagation velocity greater than the speed of light, Poincaré observed, would be "difficult to understand," because attraction would then be a function of a position in space not yet occupied by the active mass (p. 167).

These were not the only conditions Poincaré wanted to satisfy. The new law of gravitation had also (1) to behave in the same way as electromagnetic forces under a Lorentz transformation, (2) to reduce to Newton's law in the case of relative rest of the two bodies, and (3) to come as close as possible to Newton's law in the case of small velocities. Posed in this way, Poincaré noted, the problem remains indeterminate, save in the case of null relative velocity, where the propagation velocity of gravitation does not enter into consideration. Poincaré reasoned that if two bodies have a common rectilinear velocity, then the force on the passive mass is orthogonal to an ellipsoid, at the center of which lies the active mass.

Undeterred by the indeterminacy of the question in general, Poincaré set about identifying quantities invariant with respect to the Lorentz group, from which he wanted to construct a law of gravitation satisfying the constraints just mentioned. To assist in the identification and interpretation of these invariants, Poincaré referred to a space of four dimensions. "Let us regard," he wrote,

$$\begin{array}{cccc} x, & y, & z, & t\sqrt{-1} \\ \delta x, & \delta y, & \delta z, & \delta t\sqrt{-1} \\ \delta_1 x, & \delta_1 y, & \delta_1 z, & \delta_1 t\sqrt{-1}, \end{array}$$

> as the coordinates of 3 points P, P', P'', in space of 4 dimensions. We see that the Lorentz transformation is merely a rotation in this space about the origin, regarded as fixed. Consequently, we will have no distinct invariants apart from the 6 distances between the 3 points P, P', P'', considered separately and with the origin, or, if one prefers, apart from the 2 expressions:
>
> $$x^2 + y^2 + z^2 - t^2, \qquad x\delta x + y\delta y + z\delta z - t\delta t,$$
>
> or the 4 expressions of like form deduced by arbitrary permutation of the 3 points P, P', P''.[17]

Here Poincaré formed three quadruplets representing the differential displacement of two point masses, with respect to a certain four-dimensional vector space, later called a pseudo-Euclidean space.[18] By introducing such a 4-space, Poincaré simplified the task of identifying quantities invariant with respect to the Lorentz transformations, the line interval of the new space being formally identical to that of a Euclidean 4-space. He treated his three points P, P', and P'' as 4-vectors, the scalar products of which are invariant, just as in Euclidean space. In fact, Poincaré did not employ vector terminology or notation in his study of gravitation, but provided formal definitions of certain objects later called 4-vectors.

Poincaré's habit, and that of the overwhelming majority of his French colleagues in mathematical physics well into the 1920s, was to express ordinary vector quantities in Cartesian coordinate notation, and to forgo notational shortcuts when differentiating, writing these operations out in full.[19] Although he did not exclude symbols such as Δ or $\Box$ from his scientific papers and lectures, he employed them parsimoniously.[20] In line with this practice, Poincaré did little to promote vector methods from his chair at the Sorbonne. In twenty volumes of lectures on mathematical physics and celestial mechanics, there is not a single propadeutic on quaternions or vector algebra.[21] Poincaré deplored the "long calculations rendered obscure by notational complexity" in W. Voigt's molecular theory of light, and seems to have been of the opinion that in general, new notation only burdened the reader.[22]

The point of forming quadruplets was to obtain a set of Lorentz-invariants corresponding to the ten variables entering into the right-hand side of the new force law, representing the squared distance in space and time of the two bodies and their velocities (ξ, η, ζ, ξ_1, η_1, ζ_1). How did Poincaré obtain his invariants? According to the method cited above, six invariants were to be found from the distances between P, P', P'', and the origin, or from the scalar products of P, P', and P''. These six intermediate invariants were then to be combined to obtain homogeneous invariants depending on the duration of propagation of gravitational action and the velocities of the two point masses. Poincaré skipped over the intermediate step and produced the following four invariants, in terms of squared distance, distance and velocity (twice), and the velocity product:

$$\sum x^2 - t^2,\ \frac{t-\sum x\xi}{\sqrt{1-\sum\xi^2}},\ \frac{t-\sum x\xi_1}{\sqrt{1-\sum\xi_1^2}},\ \frac{1-\sum\xi\xi_1}{\sqrt{(1-\sum\xi^2)\,(1-\sum\xi_1^2)}}. \tag{1}$$

The Lorentz-invariance and geometric significance of these quantities are readily verified.[23] These four invariants (1), the latter three of which were labeled A, B, and C, formed the core of Poincaré's constructive approach to the law of gravitation. (For convenience, I refer to Poincaré's four invariants (1) as his "kinematic" invariants.)

Inspection of the signs of these invariants reveals an inconsistency, the reason for which is apparent once the intermediate calculations have been performed. Instead of constructing his four invariants out of scalar products, Poincaré introduced an inversion for A, B, and C.[24] This sign inconsistency had no consequence on his search for a relativistic law of gravitation, although it affected his final result, and perplexed at least one of his readers, as I will show in §3.

What Poincaré needed next for his force law was a Lorentz-invariant expression for the force itself. Up to this point, he had neither a velocity 4-vector nor a force 4-vector definition on hand. Presumably, the search for Lorentz-invariant expressions of force led him to define these 4-vectors. Earlier in his memoir (p. 135), Poincaré had determined the Lorentz transformations of force density, but now he was interested in the Lorentz transformations of force at a point. The transformations of force density:

$$X' = k(X + \varepsilon T), \quad Y' = Y, \quad Z' = Z, \quad T' = k(T + \varepsilon X), \tag{2}$$

where k is the Lorentz factor, $k = 1/\sqrt{1-\varepsilon^2}$, and ε designates frame velocity, led Poincaré to define a fourth component of force density, T, as the product of the force density vector with velocity, $T = \sum X\xi$.[25] He gave the same definition for the temporal component of force at a point: $T_1 = \sum X_1\xi$.[26] Next, dividing force density by force at a point, Poincaré obtained the charge density ρ. Ostensibly from the transformation for charge density, Poincaré singled out the Lorentz-invariant factor:[27]

$$\frac{\rho}{\rho'} = \frac{1}{k(1+\xi\varepsilon)} = \frac{\delta t}{\delta t'}. \tag{3}$$

The components of a 4-velocity vector followed from the foregoing definitions of position and force density:

> The Lorentz transformation ... will act in the same way on ξ, η, ζ, 1 as on δx, δy, δz, δt, with the difference that these expressions will be multiplied moreover by the *same* factor $\delta t/\delta t' = 1/k(1+\xi\varepsilon)$.[28]

Concerning the latter definition, Poincaré observed a formal analogy between the force and force density 4-vectors, on one hand, and the position and velocity 4-vectors, on the other hand: these pairs of vectors transform in the same way, except that one member is multiplied by $1/k(1+\xi\varepsilon)$. While this analogy may seem mathematically transparent, it merits notice, as it appears to have eluded Poincaré at first.

With these four kinematic 4-vectors in hand, Poincaré defined a fifth quadruplet Q with components of force density $(X, Y, Z, T\sqrt{-1})$. Just as in the previous case, the scalar products of his four quadruplets P, P', P'', and Q were to deliver four new Lorentz-invariants in terms of the force acting on the passive mass (X_1, Y_1, Z_1):[29]

$$\frac{\sum X_1^2 - T_1^2}{1-\sum\xi^2}, \quad \frac{\sum X_1 x - T_1 t}{\sqrt{1-\sum\xi^2}}, \quad \frac{\sum X_1\xi_1 - T_1}{\sqrt{1-\sum\xi^2}\sqrt{1-\sum\xi_1^2}}, \quad \frac{\sum X_1\xi - T_1}{1-\sum\xi^2}. \tag{4}$$

The fourth invariant in (4) was always null by definition of T_1, leaving only three invariants, denoted M, N, and P. (In order to distinguish these invariants from the kinematic invariants, I will refer to (4) as Poincaré's "force" invariants.)

Comparing the signs of the kinematic invariants (1) with those of the force invariants (4), we see that Poincaré obtained consistent signs only for the latter invariants. He must not have computed his force invariants in the same way as his kinematic invariants, for reasons that remain obscure. It is not entirely unlikely that in the course of his analysis of the transformations of velocity and force, Poincaré realized that he could compute the force invariants directly from the scalar products of four 4-vectors. Two facts, however, argue against this reading. In the first place, Poincaré did not mention that his force invariants were the scalar products of position, velocity and force 4-vectors. Secondly, he did not alter the signs of his kinematic invariants to make them correspond to scalar products of position and velocity 4-vectors.[30] The fact that Poincaré's kinematic invariants differ from products of 4-position and 4-velocity vectors leads us to believe that when forming these invariants he was *not* thinking in terms of 4-vectors.[31]

From this point on, Poincaré worked exclusively with arithmetic combinations of three force invariants (M, N, P) and four kinematic invariants ($\sum x^2 - t^2$, A, B, C) in order to come up with a relativistic law of gravitation. He had no further use, in particular, for the four quadruplets he had identified in the process of constructing these same invariants (corresponding to modern 4-position, 4-velocity, 4-force-density and 4-force vectors), although in the end he expressed his laws of gravitation in terms of 4-force components.

To find a law applicable to the general case of two bodies in relative motion, Poincaré introduced constraints and approximations designed to reduce the complexity of his seven invariants and recover the form of the Newtonian law in the limit of slow motion ($\xi_1 \ll 1$). Poincaré naturally looked first to the velocity of propagation of gravitation. He briefly considered an emission theory, where the velocity of gravitation depends on the velocity of the source. Although the emission hypothesis was compatible with his invariants, Poincaré rejected this option because it violated his initial injunction barring a hyperlight velocity of gravitational propagation.[32] That left him with a propagation velocity of gravitation less than or equal to that of light, and to simplify his invariants Poincaré set it equal to that of light in empty space, such that $t = -\sqrt{\sum x^2} = -r$. This stipulation reduced the total number of invariants from seven to six.

With the propagation velocity of gravitation decided, Poincaré proceeded to construct a force law for point masses. He tried two approaches, the first of which is the most general. The basic idea of both approaches is to neglect terms in the square of velocity occurring in the invariants, and to compare the resulting approximations with their Newtonian counterparts. In the Newtonian scheme, the coordinates of the active mass point differ from those in the relativistic scheme (cf. note 18); Poincaré took the former to be $(x_0 + x_1, y_0 + y_1, z_0 + z_1)$ at the instant of time t_0, where the subscript 0 corresponds to the position of the passive mass point, and the coordinates with subscript 1 are found by assuming uniform motion of the source:

$$x = x_1 - \xi_1 r,\ y = y_1 - \eta_1 r,\ z = z_1 - \zeta_1 r,\ r = r_1 - \sum x\xi_1. \tag{5}$$

In the first approach, Poincaré made use of both the kinematic and force invariants. Substituting the values (5) into the kinematic invariants A, B, and C from (1) and the force invariants M, N, and P from (4), neglecting terms in the square of velocity, Poincaré obtained their sought-after Newtonian counterparts. Replacing the force vector occurring in the transformed force invariants by Newton's law ($\sum X_1 = -1/r_1^2$), and rearranging, Poincaré obtained three quantities in terms of distance and velocity.[33] He then re-expressed these quantities in terms of two of his original kinematic invariants, A and B, and equated the three resulting kinematic invariants to their corresponding original force invariants (4). He now had the solution in hand; three expressions relate his force invariants (containing the force vector $\sum X_1$) to two of his kinematic invariants:

$$M = \frac{1}{B^4}, \qquad N = \frac{+A}{B^2}, \qquad P = \frac{A - B}{B^3}. \tag{6}$$

He noted that complementary terms could be entertained for the three relations (6), provided that they were certain functions of his kinematic invariants A, B, and C. Then without warning, he cut short his demonstration, remarking that the gravitational force components would take on imaginary values:

> The solution (6) appears at first to be the simplest, nonetheless, it may not be adopted. In fact, since M, N, P are functions of X_1, Y_1, Z_1, and $T_1 = \Sigma X_1 \xi$, the values of X_1, Y_1, Z_1 can be drawn from these three equations (6), but in certain cases these values would become imaginary.[34]

The quoted remark seems to suggest that for selected values of the particle velocities, the force turns out to be imaginary. However, the real difficulty springs from the equation $M = 1/B^4$, which allows for a repulsive force. The general approach failed to deliver.[35]

The fact that Poincaré published the preceding derivation may be understood in one of two ways. On the one hand, there is a psychological explanation: Poincaré's habit, much deplored by his peers, was to present his findings more or less in the order in which he found them. The case at hand may be no different from the others. On the other hand, Poincaré may have felt it worthwhile to show that the general approach breaks down. From the latter point of view, Poincaré's result is a positive one.

For his second attack on the law of gravitation, Poincaré adopted a less general approach. He knew where his first approach had become unsuitable, and consequently, leaving aside his three force invariants, he fell back on the form of his basic force 4-vector, which he now wrote in terms of his kinematic invariants, re-expressed in terms of $r = -t$, $k_0 = 1/\sqrt{1-\xi^2}$, and $k_1 = 1/\sqrt{1-\xi_1^2}$.[36] He assumed the gravitational force on the passive mass (moving with velocity ξ, η, ζ) to be a function of the distance separating the two mass points, the velocity of the passive mass point, and the velocity of the source, with the form:

$$
\begin{aligned}
X_1 &= x\frac{\alpha}{k_0} + \xi\beta + \xi_1\frac{k_1}{k_0}\gamma, \qquad & Z_1 &= z\frac{\alpha}{k_0} + \zeta\beta + \zeta_1\frac{k_1}{k_0}\gamma, \\
Y_1 &= y\frac{\alpha}{k_0} + \eta\beta + \eta_1\frac{k_1}{k_0}\gamma, \qquad & T_1 &= -r\frac{\alpha}{k_0} + \beta + \frac{k_1}{k_0}\gamma,
\end{aligned}
\tag{7}
$$

where α, β, and γ denote functions of the kinematic invariants.[37] By definition, the component T_1 is the scalar product of the ordinary force and the velocity of the passive mass point, $T_1 = \sum X_1\xi$, such that the three functions α, β, γ satisfy the equation:

$$
-A\alpha - \beta - C\gamma = 0. \tag{8}
$$

Poincaré further assumed $\beta = 0$, thereby eliminating a term depending on the velocity of the passive mass, and fixing the value of γ in terms of α. Applying the same slow-motion approximation and translation (5) as in his initial approach, Poincaré found $X_1 = \alpha x_1$, and by comparison with Newton's law, α reduces to $-1/r_1^3$. In terms of the kinematic invariants (1), this relation was expressed as $\alpha = 1/B^3$, and the law of gravitation (7) took on the form:[38]

$$
\begin{aligned}
X_1 &= \frac{x}{k_0B^3} - \xi_1\frac{k_1}{k_0}\frac{A}{B^3C}, \qquad & Z_1 &= \frac{z}{k_0B^3} - \zeta_1\frac{k_1}{k_0}\frac{A}{B^3C}, \\
Y_1 &= \frac{y}{k_0B^3} - \eta_1\frac{k_1}{k_0}\frac{A}{B^3C}, \qquad & T_1 &= -\frac{r}{k_0B^3} - \frac{k_1}{k_0}\frac{A}{B^3C}.
\end{aligned}
\tag{9}
$$

Inspection of Poincaré's gravitational force (9) reveals two components: one parallel to the position 4-vector between the passive mass and the retarded source, and one parallel to the source 4-velocity. The law was not unique, Poincaré noted, and it neglected possible terms in the velocity of the passive mass.

Poincaré underlined the open-ended nature of his solution by proposing a second gravitational force law. Rearranging (9) and replacing the factor $1/B^3$ by C/B^3, such that the force depended linearly on the velocity of the passive mass, Poincaré arrived at a second law of gravitation:[39]

$$
\begin{aligned}
X_1 &= \frac{\lambda}{B^3} + \frac{\eta\nu' - \zeta\mu'}{B^3}, \\
Y_1 &= \frac{\mu}{B^3} + \frac{\zeta\lambda' - \xi\nu'}{B^3}, \\
Z_1 &= \frac{\nu}{B^3} + \frac{\xi\mu' - \eta\lambda'}{B^3},
\end{aligned}
\tag{10}
$$

where

$$k_1(x + r\xi_1) = \lambda, \qquad k_1(y + r\eta_1) = \mu, \qquad k_1(z + r\zeta_1) = \nu,$$
$$k_1(\eta_1 z - \zeta_1 y) = \lambda', \quad k_1(\zeta_1 x - \xi_1 z) = \mu', \quad k_1(\xi_1 y - x\eta_1) = \nu'.$$

Poincaré neglected to write down the expression for T_1, probably because of its complicated form. (For the sake of simplicity, I refer to (9) and (10) including the latter's neglected fourth component, as Poincaré's first and second law.) The unprimed triplet $B^{-3}(\lambda, \mu, \nu)$ supports what Poincaré termed a "vague analogy" with the mechanical force on a charged particle due to an electric field, while the primed triplet $B^{-3}(\lambda', \mu', \nu')$ supports an analogy to the mechanical force on a charged particle due to a magnetic field. He identified the fields as follows:

> Now λ, μ, ν, or $\frac{\lambda}{B^3}, \frac{\mu}{B^3}, \frac{\nu}{B^3}$, is an electric field of sorts, while λ', μ', ν', or rather $\frac{\lambda'}{B^3}, \frac{\mu'}{B^3}, \frac{\nu'}{B^3}$, is a magnetic field of sorts.[40]

While Poincaré wrote freely of a "gravity wave" (*onde gravifique*), he abstained from speculating on the nature of the field referred to here. As one of the first theorists (with FitzGerald and Lorentz) to have employed retarded potentials in Maxwellian electrodynamics, Poincaré must have considered the possibility of introducing a corresponding gravitational 4-potential.[41] But as matters stood when Poincaré submitted this paper for publication in July, 1905, he was not in a position to elaborate the physics of fields in four-dimensional terms, since he possessed neither a 4-potential nor a 6-vector.

Poincaré had realized the objective of formulating a Lorentz-invariant force of gravitation. As we have seen, he surpassed this objective by identifying not one but two such force laws. Designed to reduce to Newton's law in the first order of approximation in ξ_1 (or particle velocity divided by the speed of light), Poincaré's laws could diverge from Newton's only in second-order terms. The argument satisfied Poincaré, who did not report any precise numerical results, explaining that this would require further investigation. Instead, he noted that the disagreement would be ten thousand times smaller than a first-order difference stemming from the assumption of a propagation velocity of gravitation equal to that of light, "*ceteris non mutatis*" (p. 175). His result contradicted Laplace, who had predicted an observable first-order effect arising from just such an assumption. At the very least, Poincaré had demonstrated that Laplace's argument was not compelling in the context of the new dynamics.[42]

On several occasions over the next seven years, Poincaré returned to the question of gravitation and relativity, without ever comparing the predictions of his laws with observation. During his 1906–1907 Sorbonne lectures, for example, when he developed a general formula for perihelion advance, Poincaré used a Lagrangian approach, rather than one or the other of his laws (Poincaré 1953, 238). Student notes of this course indicate that he stopped short of a numerical evaluation for the various electron models (perhaps leaving this as an exercise). However, Poincaré later provided the relevant numbers in a general review of electron theory. Lorentz's theory called for an extra 7" centennial advance by Mercury's perihelion, a figure slightly greater than the one for Abraham's non-relativistic electron theory.[43] According to the best available data, Mercury's anomalous perihelial advance was 42", prompting Poincaré to remark that another explanation would have to be found in order to account for the remaining seconds of arc. Astronomical observations, Poincaré concluded soberly, provided no arguments in favor of the new electron dynamics.[44]

Poincaré capsulized the situation of his new theory in a fable in which Lorentz plays the role of Ptolemy, and Poincaré that of an unknown astronomer appearing sometime between Ptolemy and Copernicus. The unknown astronomer notices that all the planets traverse either an epicycle or a deferent in the same lapse of time, a regularity later captured in Kepler's second law. The analogy to electron dynamics turns on a regularity discovered by Poincaré in his study of gravitation:

> If we were to admit the postulate of relativity, we would find the same number in the law of gravitation and the laws of electromagnetism, which would be the velocity of light; and we would find it again in all the other forces of any origin whatsoever.[45]

This common propagation velocity of gravitational action, of electromagnetic fields, and of any other force, could be understood in one of two ways:

> Either everything in the universe would be of electromagnetic origin, or this aspect–shared, as it were, by all physical phenomena–would be a mere epiphenomenon, something due to our methods of measurement.[46]

If the electromagnetic worldview were valid, all particle interactions would be governed by Maxwell's equations, featuring a constant propagation velocity. Otherwise, the common propagation velocity of forces had to be a result of a measurement convention. In relativity theory, as Poincaré went on to point out, the measurement convention to adopt was one defining lengths as equal if and only if spanned by a light signal in the same lapse of time, as this convention was compatible with the Lorentz contraction. There was a choice to be made between the electromagnetic worldview (as realized in the electron models of Abraham and Bucherer-Langevin) and the postulate of relativity (as upheld by the Lorentz-Poincaré electron theory). Although Poincaré favored the latter theory, he felt that its destiny was to be superseded, just as Ptolemaic astronomy was superseded by Copernican heliocentrism.

The failure of his Lorentz-invariant law of gravitation to explain the anomalous advance of Mercury's perihelion probably fed Poincaré's dissatisfaction with the Lorentz-Poincaré theory in general, but what he found particularly troubling at the time was something else altogether: the discovery of magneto-cathode rays. There is no place in the Lorentz-Poincaré electron theory for rays that are both neutral (as Paul Villard reported in June, 1904) and deflected by electric and magnetic fields, which is probably why Poincaré felt the "entire theory" to be "endangered" by magneto-cathode rays.[47]

Uncertainty over the empirical adequacy of the Lorentz-Poincaré electron theory may explain why the *Rendiconti* memoir was Poincaré's last in the field of electron physics. But is it enough to explain his disinterest in the development of a four-dimensional formalism? One year after the publication of his article on electron dynamics, Poincaré commented:

> A translation of our physics into the language of four-dimensional geometry does in fact appear to be possible; the pursuit of this translation would entail great pain for limited profit, and I will just cite Hertz's mechanics, where we see something analogous. Meanwhile, it seems that the translation would remain less simple than the text and would always have the feel of a translation, and that three-dimensional language seems the best suited to the description of our world, even if one admits that this description may be carried out in another idiom.[48]

Poincaré clearly saw in his own work the outline of a four-dimensional formalism for physics, yet he saw no future in its development, and this, entirely apart from the question of the empirical adequacy of the Lorentz-Poincaré theory.

Why did Poincaré discount the value of a language tailor-made for relativity? Three sources of disinterest in such a prospect spring to mind, the first of which stems from his conventionalist philosophy of science. Poincaré recognized an important role for notation in the exact sciences, as he famously remarked with respect to Edmond Laguerre's work on quadratic forms and Abelian functions that

> in the mathematical sciences, having the right notation is philosophically as important as having the right classification in the life sciences.[49]

More than likely, Poincaré was aware of the philosophical implications of a four-dimensional notation for physics, although he had yet to make his views public. But given his strong belief in the immanence of Euclidean geometry's fitness for physics, he must have considered the chances for success of such a language to be vanishingly small.[50]

A second source for Poincaré's disinterest in four-dimensional formalism is his practice of physics. As mentioned above, Poincaré dispensed with vectorial systems (and most notational shortcuts); he even avoided writing "i" for $\sqrt{-1}$. When considered in conjunction with his conventionalist belief in the suitability of Euclidean geometry for physics, this conservative habit with respect to notation makes Poincaré appear all the less likely to embrace a four-dimensional language for physics.

The third possible source of discontent is Poincaré's vexing experience with invariants of pseudo-Euclidean 4-space. As shown above(p. 11), Poincaré's first approach to the construction of a law of gravitation ended unsatisfactorily, and the failure of Poincaré's intuition in this instance may well have colored his view of the prospects for a four-dimensional physics.

An immediate consequence of Poincaré's refusal to work out the form of four-dimensional physics was that others could readily pick up where he left off. Roberto Marcolongo (1862–1945), Professor of Mathematical Physics in Messina, and a leading proponent of vectorial analysis, quickly discerned in Poincaré's paper a potential for formal development. Marcolongo referred, like Poincaré, to a four-dimensional space with one imaginary axis, but defined the fourth coordinate as the product of time t and the negative square root of -1 (i.e., $-t\sqrt{-1}$ instead of $t\sqrt{-1}$). After forming a 4-vector potential out of the ordinary vector and scalar potentials, and defining a 4-current vector, he expressed the Lorentz-covariance of the equations of electrodynamics in matrix form. No other applications were forthcoming from Marcolongo, and a failure to produce further 4-vector quantities and functions limited the scope of his contribution, which went unnoticed outside of Italy.[51] Nothing further on Poincaré's method appeared in print until April, 1908, when Hermann Minkowski's paper on the four-dimensional formalism and its application to the problem of gravitation appeared in the *Göttinger Nachrichten.*

2 Hermann Minkowski's spacetime laws of gravitation

The young Hermann Minkowski, fifth child of an immigrant family of Russian Jews, attended the Altstädtische Gymnasium in Königsberg (later Kaliningrad). Shortly after graduation, Minkowski submitted an essay for the Paris Academy's 1882 Grand Prize in Mathematical Sciences. His entry on quadratic forms shared top honors with a submission by the seasoned British mathematician Henry J. S. Smith, his senior by thirty-eight years.[52] The young mathematician went on to study with Heinrich Weber in Königsberg, and with Karl Weierstrass and Leopold

Kronecker in Berlin. In the years following the prize competition, Minkowski became acquainted with Poincaré's writings on algebraic number theory and quadratic forms, and in particular, with a paper in Crelle's *Journal* containing some of the results from Minkowski's prize paper, still in press. To his friend David Hilbert he confided the "angst and alarm" brought on by Poincaré's entry into his field of predilection; with his "swift and versatile" energy, Poincaré was bound to bring the whole field to closure, or so it seemed to him at the time.[53] From the earliest, formative years of his scientific career, Minkowski found in Poincaré–his senior by a decade–a daunting intellectual rival.

While Minkowski had discovered in Poincaré a rival, he was soon to find that that the Frenchman could also be a teacher, from whom he could learn new analytical skills and methods. Named Privatdozent in Bonn in 1887, Minkowski contributed to the abstract journal *Jahrbuch der Fortschritte der Mathematik*, and in 1892, took on the considerable task of summarizing the results of the paper for which Poincaré was awarded the King Oscar II Prize (Minkowski 1890). The mathematics Poincaré created in his prize paper (the study of homoclinic points in particular) was highly innovative, and at the same time, difficult to follow. Among those whom we know had trouble understanding certain points of Poincaré's prize memoir were Charles Hermite, Gustav Mittag-Leffler, and Karl Weierstrass, who happened to constitute the prize committee.[54] Minkowski, however, welcomed the review as a learning opportunity, as he wrote to his friend and former teacher, Adolf Hurwitz:

> Poincaré's prize paper is also among the works I have to report on for the *Fortschritte*. I am quite fond of it. It is a fine opportunity for me to get acquainted with problems I have not worried about too much up to now, since I will naturally set a positive goal of making my case well.[55]

In the 1890s, building on his investigations of the algebraic theory of quadratic forms, Minkowski developed the geometric analog to this theory: geometrical number theory. A high point of his efforts in this new field, and one which contributed strongly to the establishment of his reputation in mathematical circles, was the publication of *Geometrie der Zahlen* (1896). The same year, Minkowski accepted a chair at Zurich Polytechnic, whereby he rejoined Hurwitz. Minkowski's lectures on mathematics and mathematical physics attracted a small following of talented and ambitious students, including the future physicists Walter Ritz and Albert Einstein, and the budding mathematicians Marcel Grossmann and Louis Kollros.[56]

Minkowski's lectures on mechanics in Zurich throw an interesting light on his view of symbolic methods in physics at the close of the nineteenth century.

The theory of quaternions, he noted in 1897, was used nowhere outside of England, due to its "relatively abstract character and inherent difficulty."[57] Two of its fundamental concepts, scalars and vectors, had nevertheless gained broad approval among physicists, Minkowski wrote, and had found "frequent application especially in the theory of electricity."[58] Applications of quaternions to problems of physics were advanced in Germany with the publication of Felix Klein and Arnold Sommerfeld's *Theorie des Kreisels*, a work referred to in Minkowski's lecture notes of 1898–1899.[59] Minkowski admired Klein and Sommerfeld's text, expressing "great interest" in the latter to Sommerfeld, along with his approval of the fundamental significance accorded to the concept of momentum. However, their text did not make the required reading list for Minkowski's course in mechanics.[60]

In 1899, at the request of Sommerfeld, who a year earlier had agreed to edit the physics volumes of Felix Klein's ambitious *Encyclopedia of the Mathematical Sciences including Applications* (hereafter *Encyklopädie*), Minkowski agreed to cover a topic in molecular physics he knew little about, but one perfectly suited to his skills as an analyst: capillarity.[61] The article that appeared seven years later represented his second contribution to physics, after a short note on theoretical hydrodynamics published in 1888, but which, ten years later, Minkowski claimed no one had read–save the abstracter.[62]

When Minkowski accepted Göttingen's newly-created third chair of pure mathematics in the fall of 1902, the pace of his research changed brusquely. The University of Göttingen at the turn of the last century was a magnet for talented young mathematicians and physicists.[63] Minkowski soon was immersed in the activities of Göttingen's Royal Society of Science, its mathematical society, and research seminars. Several faculty members, including Max Abraham, Gustav Herglotz, Eduard Riecke, Karl Schwarzschild, and Emil Wiechert, actively pursued theoretical or experimental investigations motivated by the theory of electrons, and it was not long before Minkowski, too, took up the theory. During the summer semester of 1905 he co-led a seminar with Hilbert on electron theory, featuring reports by Wiechert and Herglotz, and by Max Laue, who had just finished a doctoral thesis under Max Planck's supervision.[64]

Along with seminars on advanced topics in physics and analytical mechanics, Göttingen featured a lively mathematical society, with weekly meetings devoted to presentations of work-in-progress and reports on scientific activity outside of Göttingen. The electron theory was a frequent topic of discussion in this venue. For instance, the problem of gravitational attraction was first addressed by Schwarzschild in December, 1904, in a report on Alexander Wilkens' recent paper on the compatibility of Lorentz's electron theory with astronomical observations.[65]

A focal point of sorts for the mathematical society, Poincaré's scientific output fascinated Göttingen scientists in general, and Minkowski in particular, as mentioned above.[66] Minkowski reported to the mathematical society on Poincaré's publications on topology, automorphic functions, and capillarity, devoting three talks in 1905–1906 to Poincaré's 1888–1889 Sorbonne lectures on this subject (Poincaré 1895). Others reporting on Poincaré's work were Conrad Müller on Poincaré's St. Louis lecture on the current state and future of mathematical physics (31 January, 1905), Hugo Broggi on probability (27 October, 1905), Ernst Zermelo on a boundary-value problem (12 December, 1905), Erhard Schmidt on the theory of differential equations (19 December, 1905), Max Abraham on the Sorbonne lectures (6 February, 1906) and Paul Koebe on the uniformization theorem (19 November, 1907). One gathers from this list that the Göttingen mathematical society paid attention to Poincaré's contributions to celestial mechanics, mathematical physics, and pure mathematics, all subjects intersecting with the ongoing research of its members. It also appears that no other member of the mathematical society was quite as assiduous in this respect as Minkowski.[67]

When Einstein's relativity paper appeared in late September, 1905, it drew the attention of the Bonn experimentalist Walter Kaufmann, a former Göttingen Privatdozent and friend of Max Abraham, but neither Abraham nor any of his colleagues rushed to report on the new ideas to the mathematical society.[68] Poincaré's long memoir on the dynamics of the electron, published in January, 1906, fared better, although nearly two years went by before Minkowski found an occasion to comment on Poincaré's gravitation theory, and to present his own related work-in-progress. Minkowski's typescript has been conserved, and is the source referred to here.[69]

On the occasion of the 5 November, 1907, meeting of the mathematical society, Minkowski began his review of Poincaré's work by observing that gravitation remained an "important question" in relativity theory, since it was not yet known "how the law of gravitation is arranged for in the realm of the principle of relativity."[70] The basic problem of gravitation and relativity, in other words, had not been solved by Poincaré. Eliding mention of Poincaré's two laws, Minkowski recognized in his work only one positive result: by considering gravitational attraction as a "pure mathematical problem," he said, Poincaré had found gravitation to propagate with the speed of light, thereby overturning the standard Laplacian argument to the contrary.[71] Minkowski expressed dissatisfaction with Poincaré's approach, allowing that Poincaré's was "only one of many" possible laws, a fact stemming from its construction out of Lorentz-invariants. Consequently, Poincaré's investigation "had by no means a definitive character."[72] A critical remark of this sort often introduces an alternative theory, but in this instance none was forthcoming, and as I will show in what follows, there is ample reason to doubt that Minkowski was actually in a position to improve on Poincaré's investigation. Nonetheless, at the end of his talk Minkowski set forth the possibility of elaborating his report.

Minkowski's lecture was not devoted entirely to Poincaré's investigation of Lorentz-invariant gravitation. The purpose of his lecture, according to the published abstract, was to present a new form of the equations of electrodynamics leading to a mathematical redescription of physical laws in four areas: electricity, matter, mechanics, and gravitation.[73] These laws were to be expressed in terms of the differential equations used by Lorentz as the foundation of his successful theory of electrons (1904a), but in a form taking greater advantage of the invariance of the quadratic form $x^2 + y^2 + z^2 - c^2t^2$. Physical laws, Minkowski stated, were to be expressed with respect to a four-dimensional manifold, with coordinates x_1, x_2, x_3, x_4, where units were chosen such that $c = 1$, the ordinary Cartesian coordinates x, y, and z, went over into the first three, and the fourth was defined to be an imaginary time coordinate, $x_4 = it$. Implicitly, then, Minkowski took as his starting point the four-dimensional vector space described in the last section of Poincaré's memoir on the dynamics of the electron.

Minkowski acknowledged, albeit obliquely, a certain continuity between Poincaré's memoir and his own program to reform the laws of physics in four-dimensional terms. By formulating the electromagnetic field equations in four-dimensional notation, Minkowski said he was revealing a symmetry not realized by his predecessors, not even by Poincaré himself (Walter 1999b, 98). While Poincaré had not sought to modify the standard form of Maxwell's equations, Minkowski felt it was time for a change. The advantage of expressing Maxwell's equations in the new notation, Minkowski informed his Göttingen colleagues, was that they were then "easier to grasp" (p. 11).

His reformulation naturally began in the electromagnetic domain, with an expression for the potentials. He formed a 4-vector potential denoted (ψ) by taking the ordinary vector potential over for the first three components, and setting the fourth component equal to the product of i and the scalar potential. The same method was applied to obtain a four-component quantity for current density: for the first three components, Minkowski took over the convection current density vector, $\varrho\mathfrak{w}$, or charge density times velocity, and defined the fourth component to be the product of i and the charge density. Rewriting the potential and current density vectors in this way, Minkowski imposed what is now known as the Lorenz condition, $\mathrm{Div}(\psi) = 0$, where Div is an extension of ordinary divergence. This led him to the following expression, summarizing two of the four Maxwell equations:

$$\Box\psi_j = -\varrho_j \quad (j = 1,\ 2,\ 3,\ 4), \tag{11}$$

where $\Box$ is the d'Alembertian, employed earlier by Poincaré (cf. note 20).

Of the formal innovations presented by Minkowski to the mathematical society, the most remarkable was what he called a *Traktor*, a six-component entity used to represent the electromagnetic field.[74] He defined the six components via the 4-vector potential, using a two-index notation: $\psi_{jk} = \partial\psi_k/\partial x_j - \partial\psi_j/\partial x_k$, noting the antisymmetry relation $\psi_{kj} = -\psi_{jk}$, and zeros along the diagonal $\psi_{jj} = 0$. In this way, the Traktor components ψ_{14}, ψ_{24}, ψ_{34}, ψ_{23}, ψ_{31}, ψ_{12} match up with the field quantities $-i\mathfrak{E}_x$, $-i\mathfrak{E}_y$, $-i\mathfrak{E}_z$, $\mathfrak{h}_x$, $\mathfrak{h}_y$, $\mathfrak{h}_z$.[75]

The Traktor first found application when Minkowski turned to his second topic: the four-dimensional view of matter. Ignoring the electron theories of matter of Lorentz and Joseph Larmor, Minkowski focused uniquely on the macroscopic electrodynamics of moving media.[76] For this subject he introduced a "Polarisationstraktor", (p), along with a 4-current-density, (σ), defined by the current density vector $\mathbf{i}$ and the charge density ϱ: $(\sigma) = i_x, i_y, i_z, \mathrm{i}\varrho$ (typescript, p. 9). Recalling (11), Minkowski wrote Maxwell's source equations in covariant form:

$$\frac{\partial p_{1j}}{\partial x_1} + \frac{\partial p_{2j}}{\partial x_2} + \frac{\partial p_{3j}}{\partial x_3} + \frac{\partial p_{4j}}{\partial x_4} = \sigma_j - \varrho_j. \tag{12}$$

Minkowski's relativistic extension of Maxwell's theory was all the simpler in that it elided the covariant expression of the constitutive equations, which involves 4-velocity.[77] While none of his formulas invoked 4-velocity, Minkowski acknowledged that his theory required a "velocity vector of matter $(w) = w_1, w_2, w_3, w_4$" (typescript, p. 10).

In order to express the "visible velocity of matter in any location," Minkowski needed a new vector as a function of the coordinates x, y, z, t (typescript, p. 7). Had he understood Poincaré's 4-velocity definition(above, p. 9), he undoubtedly would have employed it at this point. Instead, following the same method of generalization from three to four components successfully applied in the case of 4-vector potential, 4-current density, and 4-force density, Minkowski took over the components of the velocity vector $\mathfrak{w}$ for the spatial elements of the quadruplet designated w_1, w_2, w_3, w_4:

$$\mathfrak{w}_x, \quad \mathfrak{w}_y, \quad \mathfrak{w}_z, \quad \mathrm{i}\sqrt{1-\mathfrak{w}^2}. \tag{13}$$

There are two curious aspects to Minkowski's definition. First of all, its squared magnitude does not vanish when ordinary velocity vanishes; even a particle at rest with respect to a reference frame is described in that frame by a 4-velocity vector of nonzero length. This is also true of Poincaré's 4-velocity definition, and is a feature of relativistic kinematics. Secondly, the components of Minkowski's quadruplet do not transform like the coordinates x_1, x_2, x_3, x_4, and consequently lack what he knew to be an essential property of a 4-vector.[78]

The most likely source for Minkowski's blunder is Poincaré's paper. We recall that Poincaré's derivation of his kinematic invariants ignored 4-vectors(above, p. 8), and what is more, his paper features a misleading misprint, according to which the spatial part of a 4-velocity vector is given to be the ordinary velocity vector.[79] Other sources of error can easily be imagined, of course.[80] It is strange that Minkowski did not check the transformation properties of his 4-velocity definition, but given its provenance, he probably had no reason to doubt its soundness.

Minkowski's mistake strongly suggests that at the time of his lecture, he did not yet conceive of particle motion in terms of a worldline parameter. Such an approach to particle motion would undoubtedly have spared Minkowski the error, since it renders trivial the definition of 4-velocity.[81] As matters stood in November, 1907, however, Minkowski could proceed no further with his project of reformulation.[82] The development of four-dimensional mechanics was hobbled by Minkowski's spare stock of 4-vectors even more than that of electrodynamics. Although Minkowski defined a force-density 4-vector, the fourth component of which he correctly identified as the energy equation, he did not go on to define 4-force at a point.[83] Once again, the definition of a force 4-vector at a point would have been trivial, had Minkowski possessed a correct 4-velocity definition. No more than a review of Planck's recent investigation (Planck 1907), Minkowski's discussion of mechanics involved no 4-vectors at all. Likewise for the subsequent section on gravitation, which reviewed Poincaré's theory, as shown above(p. 19).Without a valid 4-vector for velocity, Minkowski's electrodynamics of moving media was severely hobbled; without a point force 4-vector, his four-dimensional mechanics and theory of gravitation could go nowhere.

The difficulty encountered by Minkowski in formulating a four-dimensional approach to physics is surprising in light of the account he gave later of the background to his discovery of spacetime (Minkowski 1909). Minkowski presented his spacetime view of relativity theory as a simple application of group methods to the differential equations of classical mechanics. These equations were known to be invariant with respect to uniform translations, just as the squared sum of differentials $dx^2 + dy^2 + dz^2$ was known to be invariant with respect to rotations and translations of Cartesian axes in Euclidean 3-space, and yet no one, he said, had thought of compounding the two corresponding transformation groups. When this is done properly (by introducing a positive parameter c), one ends up with a group Minkowski designated G_c, with respect to which the laws of physics are covariant. (The group G_c is now known as the Poincaré group.) Presumably, the four-dimensional approach appeared simple to Minkowski in hindsight, several months after his struggle with 4-velocity.

In summary, while Minkowski formulated the idea of a four-dimensional language for physics based on the form-invariance of the Maxwell equations under the transformations of the Lorentz group, his development of this program beyond electrodynamics was hindered by a misunderstanding of the four-dimensional counterpart of an ordinary velocity vector. This was to be only a temporary obstacle.

On 21 December, 1907, Minkowski presented to the Royal Society of Science in Göttingen a memoir entitled "The Basic Equations for Electromagnetic Processes in Moving Bodies," which I will refer to for brevity as the *Grundgleichungen*.[84] Minkowski's memoir revisits in detail most of the topics introduced in his 5 November lecture to the mathematical society, but employs none of the jargon of spaces, geometries, and manifolds. What it emphasizes instead–in agreement with its title–is the achievement of the first theory of electrodynamics of moving bodies in full conformance to the principle of relativity. Also underlined is a second result described as "very surprising": the laws of mechanics follow from the postulate of relativity and the law of energy conservation alone. On the four-dimensional world and the new form of the equations of electrodynamics, both topics headlined in his November lecture, Minkowski remained coy. Curiously, the introduction mentions nothing about a new formalism, even though all but one of fourteen sections introduce and employ new notation or calculation rules (not counting the appendix).

The added emphasis on the laws of mechanics in Minkowski's introduction, on the other hand, reflects Minkowski's recent discovery of correct definitions of 4-velocity and 4-force, along with geometric interpretations of these entities. It was in the *Grundgleichungen* that Minkowski first employed the term "spacetime" (*Raumzeit*).[85] For example, he introduced 4-current density as the exemplar of a "spacetime vector of the first kind" (§ 5), and used it to derive a velocity 4-vector. Identifying ϱ_1, ϱ_2, ϱ_3, ϱ_4 with $\varrho\mathfrak{w}_x$, $\varrho\mathfrak{w}_y$, $\varrho\mathfrak{w}_z$, $i\varrho$, just as he had done in his lecture of 5 November, Minkowski wrote the transformation to a primed system moving with uniform velocity $q < 1$:

$$\varrho' = \varrho\left(\frac{-q\mathfrak{w}_z + 1}{\sqrt{1-q^2}}\right),\ \varrho'\mathfrak{w}'_{z'} = \varrho\left(\frac{\mathfrak{w}_z - q}{\sqrt{1-q^2}}\right),\ \varrho'\mathfrak{w}'_{x'} = \varrho\mathfrak{w}_x,\ \varrho'\mathfrak{w}'_{y'} = \varrho\mathfrak{w}_y. \tag{14}$$

Observing that this transformation did not alter the expression $\varrho^2(1-\mathfrak{w}^2)$, Minkowski announced an "important remark" concerning the relation of the primed to the unprimed velocity vector (§4). Dividing the 4-current density by the positive square root of the latter invariant, he obtained a valid definition of 4-velocity,

$$\frac{\mathfrak{w}_x}{\sqrt{1-\mathfrak{w}^2}},\qquad \frac{\mathfrak{w}_y}{\sqrt{1-\mathfrak{w}^2}},\qquad \frac{\mathfrak{w}_z}{\sqrt{1-\mathfrak{w}^2}},\qquad \frac{i}{\sqrt{1-\mathfrak{w}^2}}, \tag{15}$$

the squared magnitude of which is equal to -1. Minkowski seemed satisfied with this definition, naming it the spacetime velocity vector (*Raum-Zeit-Vektor Geschwindigkeit*).

The significance of the spacetime velocity vector, Minkowski observed, lies in the relation it establishes between the coordinate differentials and matter in motion, according to the expression

$$\sqrt{-(dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2)} = dt\sqrt{1-\mathfrak{w}^2}. \tag{16}$$

The Lorentz-invariance of the right-hand side of (16), signaled earlier by both Poincaré and Planck, now described the relation of the sum of the squares of the coordinate differentials to the components of 4-velocity.

The latter relation plays no direct role in Minkowski's subsequent development of the electrodynamics of moving media, and in this it is unlike the 4-velocity definition. Rewriting the right-hand side of (16) as the ratio of the coordinate differential dx_4 to the temporal component of 4-velocity, w_4, Minkowski defined the spacetime integral of (16) as the "proper time" (*Eigenzeit*) pertaining to a particle of matter. The introduction of proper time streamlined Minkowski's 4-vector expressions, for instance, 4-velocity was now expressed in terms of the coordinate differentials, the imaginary unit, and the differential of proper time, $d\tau$:

$$\frac{dx}{d\tau}, \qquad \frac{dy}{d\tau}, \qquad \frac{dz}{d\tau}, \qquad i\frac{dt}{d\tau}. \tag{17}$$

Along with the notational simplification realized by the introduction of proper time, Minkowski signaled a geometric interpretation of 4-velocity. Since proper time is the parameter of a spacetime line (or as he later called it, a worldline), it follows that 4-velocity is equal to the slope of a worldline at a given spacetime point, much like ordinary three-velocity is described by the slope of a displacement curve in classical kinematics. What Minkowski pointed out, in other words, is that 4-velocity is tangent to a worldline at a given spacetime point (p. 108).

In order to develop his mechanics, Minkowski needed a workable definition of mass. He adapted Einstein's and Planck's notion of rest mass to the arena of spacetime by considering that a particle of matter sweeps out a hypertube in spacetime. Conservation of particle mass m was then expressed as invariance of the product of rest mass density with the volume slices of successive constant-time hypersurfaces over the length of the particle's worldline, such that $dm/d\tau = 0$. Minkowski did not consider the case of variable rest mass density, which arises, for instance, in the case of heat exchange.

Minkowski's decision to adopt a constant rest mass density is linked to his view of the electrodynamics of moving media. Recall that he had introduced a six-vector in his 5 November lecture to represent the field. The product of the field and excitation six-vectors, he noted, leads to an interesting 4 by 4 matrix, combining the Maxwell stresses, Poynting vector, and electromagnetic energy density. He did not assign a name to this object, known later as the energy-momentum tensor, and often viewed as one of Minkowski's greatest achievements in electrodynamics.[86] Of special interest to Minkowski was the fact that the 4-divergence of this matrix, denoted $\operatorname{lor} S$, is a 4-vector, K:[87]

$$K = \operatorname{lor} S. \tag{18}$$

This 4-divergence (18) was used to define the "ponderomotive" force density, or generalized force per unit volume, neither mechanical nor non-mechanical in the pure sense of these terms. The 4-vector K is not normal, in general, to the velocity w of a given volume element, so to ensure that the ponderomotive force acts orthogonally to w, Minkowski added a component containing a velocity term:

$$K + (w\bar{K})w. \tag{19}$$

The parentheses in (19) indicate a scalar product, and $\bar{K}$ stands for the transpose of K. By defining the ponderomotive force density in this way, Minkowski effectively opted for an equation of motion in which 4-acceleration is normal to 4-velocity.[88] It appears that Minkowski let this view of force and acceleration guide his development of spacetime mechanics. In the latter domain, he formed a 4 by 4 matrix S in the force density and energy of an elastic media with the same transformation properties as the energy-momentum tensor S of (18), and used the 4-divergence of this tensor to express the equations of motion of a volume element of constant rest mass density ν (p. 106):

$$\nu \frac{dw_h}{d\tau} = K_h + \kappa w_h \qquad (h = 1, 2, 3, 4). \tag{20}$$

The factor κ was determined by the definition of 4-velocity to be equal to the scalar product $(K\bar{w})$, much like the definition of ponderomotive force (19). In sum, it may be supposed that the non-orthogonality with respect to a given volume element of the 4-divergence of Minkowski's asymmetric energy-momentum tensor for moving media led Minkowski to introduce a velocity term to his definition of ponderomotive force. This definition was then ported to spacetime mechanics, where for the sake of consistency, Minkowski held rest mass density constant in the equations of motion (20).

Minkowski's stipulation of constant rest mass density was eventually challenged by Max Abraham (1909, 739) and others, for reasons that do not concern us here. Despite its obvious drawbacks, it greatly simplified the tasks of outlining the mechanics of spacetime and developing a theory of gravitation. For example, it permitted him to define the equations of motion of a particle in terms of the product of rest mass and 4-acceleration, where the latter is the derivative of 4-velocity with respect to proper time. Since 4-velocity is orthogonal to 4-acceleration, for constant proper mass it is also orthogonal to a 4-vector Minkowski called a "driving force" (*bewegende Kraft*, p. 108). Minkowski wrote four equations defining this force:

$$m\frac{d}{d\tau}\frac{dx}{d\tau} = R_x,\ m\frac{d}{d\tau}\frac{dy}{d\tau} = R_y,\ m\frac{d}{d\tau}\frac{dz}{d\tau} = R_z,\ m\frac{d}{d\tau}\frac{dt}{d\tau} = R_t. \tag{21}$$

The first three expressions differ from Planck's equations of motion, in that Planck defined force as change in *momentum*, instead of mass times acceleration. It was only a few months later that Minkowski explicitly defined four-momentum as the product of 4-velocity with proper mass.[89] By dividing Minkowski's first three equations by a Lorentz factor, one obtains Planck's equations. Minkowski's fourth equation, R_t, formally dependent on the other three, expresses the law of energy conservation.[90] From energy conservation and the relativity postulate alone, Minkowski concluded, one may derive the equations of motion. This is the single "surprising" result of his investigation of relativistic mechanics, referred to at the outset of his paper (see above, p. 23).

If Minkowski found few surprises in spacetime mechanics, many of his readers were taken aback by his four-dimensional approach. For example, the first physicists to comment on his work, Albert Einstein and Jakob Laub, rewrote Minkowski's expressions in ordinary vector notation, sparing the reader the "sizable demands" (*ziemlich große Anforderungen*) of Minkowski's mathematics (Einstein 1908, 532). They did not specify the nature of the demands, but the abstracter of their paper pointed to the "special knowledge of the calculation methods" required for assimilation of Minkowski's equations.[91] In other words, for Minkowski's readers, his novel matrix calculus was the principal technical obstacle to overcome. Where Poincaré pushed rejection of formalism to an extreme, Minkowski pulled in the other direction, introducing a formalism foreign to the practice of physics. What motivated this brash move is unclear, and his choice is all the more curious because he knowingly defied the German trend of vector notation in electrodynamics.[92] As mentioned above, Minkowski was ill-disposed toward quaternions, although he admitted in print that they could be brought into use for relativity instead of matrix calculus. He spoke here from experience, as manuscript notes reveal that he used quaternions (in addition to Cartesian-coordinate representation and ordinary vector analysis) to investigate the electrodynamics of moving media.[93] In the end, however, he felt that for his purposes quaternions were "too limited and cumbersome" (*zu eng und schwerfällig*, p. 79).

As far as notation is concerned, Minkowski's treatment of differential operations broke cleanly with then-current practice. It also broke with the precedent of his 5 November lecture, where he had introduced, albeit parsimoniously, both $\Box$ and Div (see above, p. 20).For the *Grundgleichungen* he adopted a different approach, extending the ∇ to four dimensions, and labeling the resulting operator *lor,* already encountered above in (18). The name is short for Lorentz, and the effect is the operation: $\left|\partial/\partial x_1,\ \ \partial/\partial x_2,\ \ \partial/\partial x_3,\ \ \partial/\partial x_4\right|$. When applied to a 6-vector, lor results in a 4-vector, in what Minkowski described as an appropriate translation of the matrix product rule (p. 89); it also mimics the effect of the ordinary ∇. Transforming as a 4-vector, lor is liberally employed in the second part of the *Grundgleichungen*, to the exclusion of any and all particular 4-vector functions.[94] The use of lor made for a presentation of electrodynamics elegant in the extreme, at the expense of legibility for German physicists more used to thinking in terms of gradients, divergences, and curls (or rotations).

Minkowski's equations of electrodynamics departed radically in form with those of the old electrodynamics, shocking the thought patterns of physicists, and creating a phenomenon of rejection that took several years–and a new formalism–to overcome.[95] Why did Minkowski break with this tradition? Did he feel that the new physics of spacetime required a clean break with nineteenth-century practice? Perhaps, but he must have recognized that the old methods would prove resistant to change. His own subsequent practice shows as much: after writing the *Grundgleichungen* Minkowski did not bother with lor during his private explorations of the formal side of electrodynamics, preferring the coordinate method.[96]

He also relied largely–but not exclusively–on a Cartesian-coordinate approach during his preliminary investigations of the subjects treated in the *Grundgleichungen*. His surviving research notes, made up almost entirely of symbolic calculations, shed an interesting light on Minkowski's path to both a theory of the electrodynamics of moving media, and a theory of gravitation, or more generally to his process of discovery. Notably, where the subjects of mechanics and gravitation are relegated to the appendix of the *Grundgleichungen*, these notes show that Minkowski pursued questions of electrodynamics and gravitation in parallel, switching from one topic to the other three times in the course of 163 carefully numbered pages. At least fifteen of these pages are specifically concerned with gravitation; the notes are undated, but those concerning gravitation are certainly posterior to the typescript of the 5 November lecture, because unlike the latter, they feature valid definitions of 4-velocity and 4-force.

Minkowski's attempt to capture gravitational action in terms of a 4-scalar potential is of particular interest. We recall that Minkowski had expressed Maxwell's equations in terms of a 4-vector potential (11) during his lecture of 5 November, and on this basis, it was natural for him to investigate the possibility of representing gravitational force on a point mass in a fashion analogous to that of the force on a point charge moving in an electromagnetic field. In his scratch notes, Minkowski defined a 4-scalar potential Φ, in terms of which he initially devised the law of motion:

$$\begin{aligned} \frac{d}{d\tau}\frac{1}{\sqrt{1-v^2}} - \frac{\partial\Phi}{\partial t} &= 0, & \frac{d}{d\tau}\frac{-y'}{\sqrt{1-v^2}} - \frac{\partial\Phi}{\partial y} &= 0, \\ \frac{d}{d\tau}\frac{-x'}{\sqrt{1-v^2}} - \frac{\partial\Phi}{\partial x} &= 0, & \frac{d}{d\tau}\frac{-z'}{\sqrt{1-v^2}} - \frac{\partial\Phi}{\partial z} &= 0, \end{aligned} \tag{22}$$

where constants are neglected, τ denotes proper time, and primes indicate differentiation with respect to coordinate time t (i.e., $x' = dx/dt$).[97] This generalization of the Newtonian potential to a 4-scalar potential appears to be one of the first paths explored by Minkowski in his study of gravitation, but his investigation is inconclusive. In particular, there is no indication in these notes of a recognition on Minkowski's part that a four-scalar potential conflicts with the postulates of invariant rest mass and light velocity.[98] Nor is there any evidence that he considered suspending either one of these postulates.

Likewise, in the *Grundgleichungen* there is no question of adopting either a variable mass density or a gravitational 4-potential. Once he had established the foundations of spacetime mechanics, Minkowski took up the case of gravitational attraction. The problem choice is significant, in that the same question had been treated at length by Poincaré (although not to Minkowski's satisfaction, as mentioned above, p. 19).Implicitly, Minkowski encouraged readers to compare methods and results. Explicitly, proceeding in what he described (in a footnote) as a "wholly different way" from Poincaré, Minkowski said he wanted to make "plausible" the inclusion of gravitation in the scheme of relativistic mechanics (p. 109). It will become clear in what follows that his project was more ambitious than the modest elaboration of a plausibility argument, as it was designed to validate his spacetime mechanics.

The point of departure for Minkowski's theory of gravitation was quite different from that of Poincaré, because the latter's results were integrated into the former's formalism. For example, where Poincaré initially assumed a finite propagation velocity of gravitation no greater than that of light, only to opt in the end for a velocity equal to that of light, Minkowski assumed implicitly from the outset that this velocity was equal to that of light. Similarly, Poincaré initially supposed the gravitational force to be Lorentz covariant, only to opt in the end for an analog of the Lorentz force, where Minkowski required implicitly from the outset that all forces transform like the Lorentz force.

Combining geometric and symbolic arguments, Minkowski's exposition of his theory of gravitation introduces a new geometric object, the three-dimensional "ray form" (*Strahlgebilde*) of a given spacetime point, known today as a light hypercone (or lightcone). For a fixed spacetime point $B* = (x*, y*, z*, t*)$, the lightcone of $B*$ is defined by the sets of spacetime points $B = (x, y, z, t)$ satisfying the equation

$$(x - x*)^2 + (y - y*)^2 + (z - z*)^2 = (t - t*)^2, \qquad t - t* \geqq 0. \tag{23}$$

For all the spacetime points B of this lightcone, $B*$ is what Minkowski called B's lightpoint. Any worldline intersects the lightcone in one spacetime point only, Minkowski observed, such that for any spacetime point B on a worldline there exists one and only one lightpoint $B*$. Minkowski remarked in a later lecture that the lightcone divides four-dimensional space into three regions: timelike, spacelike and lightlike.[99]

Using this novel insight to the structure of four-dimensional space, in combination with the 4-vector notation set up in earlier in his memoir, Minkowski

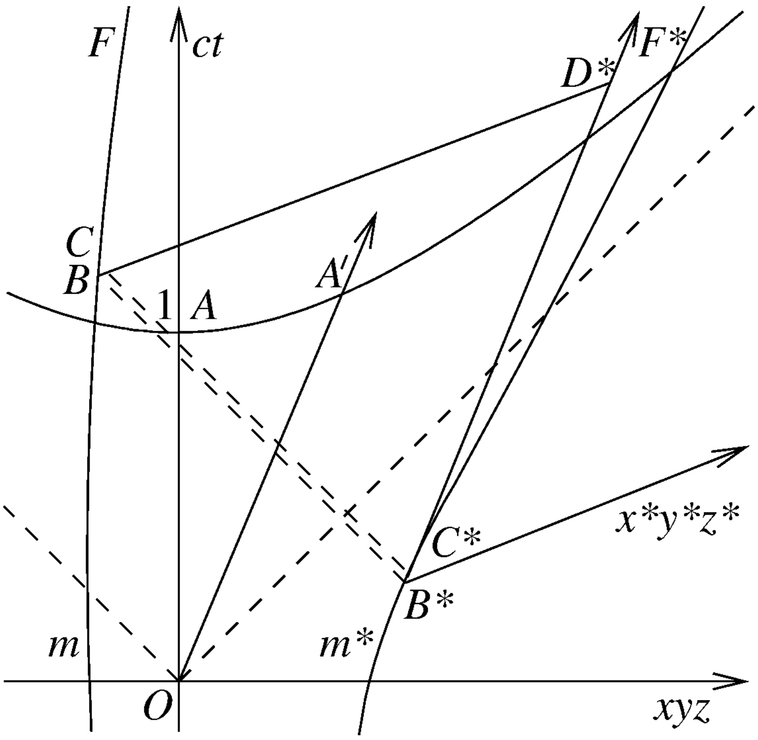

Figure 1: Minkowski's geometry of gravitation, with source in arbitrary motion.

presented and applied his law of gravitational attraction in two highly condensed pages. Minkowski's geometric argument employs non-Euclidean relations that were unfamiliar to physicists, yet he provided no diagrams. Visually-intuitive arguments had fallen into disfavor with mathematicians by this time, with the rise of the axiomatic approach to geometry favored by David Hilbert (Rowe 1997), yet Minkowski never renounced the use of figures in geometry; he employed them in earlier works on number geometry, and went on to publish spacetime diagrams in the sequel to the *Grundgleichungen*.[100] For the purposes of my reconstruction, I refer to a spacetime diagram (Figure 1) of the sort Minkowski employed in the sequel (reproduced in Figure 3).[101]

On the assumption that the force of gravitation is a 4-vector normal to the 4-velocity of the passive mass m, Minkowski derived his law of attraction in the following way. The trajectories of two particles of mass m and $m*$ correspond to two spacetime filaments F and $F*$, respectively. Minkowski's arguments refer to worldlines he called central lines (*Hauptlinien*) of these filaments, which pass through points on the successive constant-time hypersurfaces delimited by the respective particle volumes. The central lines of the filaments F and $F*$ are shown in Figure 1. An infinitesimal element of the central line of F is labeled BC, and the two lightpoints corresponding to the endpoints B and C are labeled $B*$ and $C*$ on the central line of $F*$. From the origin of the rest frame O, a 4-vector parallel to $B*C*$ intersects at A' the three-dimensional hypersurface defined by the equation $-x^2 - y^2 - z^2 + t^2 = 1$. Finally, a spacelike 4-vector $BD*$ extends from B to a point $D*$ on the worldline tangent to the central line of $F*$ at $B*$.

Referring to the latter configuration of seven spacetime points, two central lines, a lightcone and a calibration hypersurface, Minkowski expressed the spatial components of the driving force of gravitation exerted by $m*$ on m at B,

$$mm*\left(\frac{OA'}{B*D*}\right)^3 BD*. \tag{24}$$

Minkowski's gravitational driving force is composed of the latter 4-vector (24) and a second 4-vector parallel to $B*C*$ at B, such that the driving force is always orthogonal to the 4-velocity of the passive mass m at B. (For reasons of commodity, I will refer to this law of force as Minkowski's first law.)

The form of Minkowski's first law of gravitation is comparable to that of his ponderomotive force for moving media (19), in that the driving force has two components, only one of which depends on the motion of the test particle. In the gravitational case, however, Minkowski did not write out the 4-vector components in terms of matrix products. Instead, he relied on spacetime geometry and the definition of a 4-vector. The only way physicists could understand (24) was by reformulating it in terms of ordinary vectors referring to a conveniently chosen inertial frame, and even then, they could not rely on Minkowski's description alone, as it is incomplete.[102]

Even without spacetime diagrams or a transcription into ordinary vector notation, the formal analogy of (24) to Newton's law is readily apparent, and this is probably why Minkowski wrote it this way. In doing so, however, he passed up an opportunity to employ the new matrix machinery at his disposal. Had he seized this opportunity, he would have gained a simple, self-contained, coordinate-free expression of the law of gravitation, and provided readers with a more elaborate example of his calculus in action, but the latter desiderata must not have been among his primary objectives.[103]

Minkowski was not yet finished with his law of gravitation. Unlike Poincaré, after writing his law of gravitation, Minkowski went on to apply it to the particular case of uniform rectilinear motion of the source $m*$. He considered the latter in a comoving frame, in which the temporal axis is chosen to coincide with the tangent to the central line of $F*$ at $B*$ (cf. the situation described in note 102). Referring to the reconstructed spacetime diagram in Figure 2, the temporal axis is represented by a vertical line $F*$, such that the origin is established in a frame

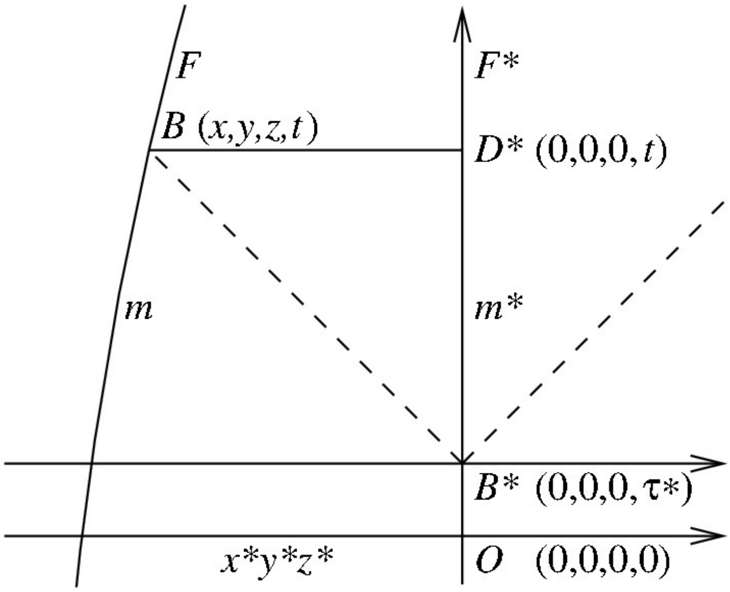

Figure 2: Minkowski's geometry of gravitation, with source in uniform motion.

comoving with m*. To the retarded position of m*, denoted B*, Minkowski assigned the coordinates (0, 0, 0, τ*), and to the position B of the passive mass m he assigned the coordinates (x, y, z, t). The geometry of this configuration fixes the location of D* at (0, 0, 0, t), from which the 4-vectors $BD* = (-x, -y, -z, 0)$ and $B*D* = (0, 0, 0, \mathrm{i}(t - \tau*))$ are determined. In this case, Minkowski pointed out, (23) reduces to:

$$x^2 + y^2 + z^2 = (t - \tau*)^2. \tag{25}$$

Substituting the above values of BD* and B*D* into Minkowski's formula (24), the spatial components of the 4-acceleration of the passive mass m at B due to the active mass m* at B* turn out to be:[104]

$$\frac{d^2x}{d\tau^2} = -\frac{m*x}{(t-\tau*)^3}, \qquad \frac{d^2y}{d\tau^2} = -\frac{m*y}{(t-\tau*)^3}, \qquad \frac{d^2z}{d\tau^2} = -\frac{m*z}{(t-\tau*)^3}. \tag{26}$$

From (26) and (25), the corresponding temporal component at B may be determined:[105]

$$\frac{d^2t}{d\tau^2} = -\frac{m*}{(t-\tau*)^2}\frac{d(t-\tau*)}{dt}. \tag{27}$$

Inspecting (26), it appears that the only difference between these acceleration components and those corresponding to Newtonian attraction is a replacement in the latter of coordinate time t by proper time τ.[106]

The formal similarity of (26) to the Newtonian law of motion under a central force probably suggested to Minkowski that his law induces Keplerian trajectories. With the knowledge gained from (26), to the effect that the only difference between classical and relativistic trajectories is that arising from the substitution of proper time for coordinate time, Minkowski demonstrated the compatibility of his relativistic law of gravitation with observation using only Kepler's equation and the definition of 4-velocity.

Writing Kepler's equation in terms of proper time yields:

$$n\tau = E - e \sin E, \tag{28}$$

where $n\tau$ denotes the mean anomaly, e the eccentricity, and E the eccentric anomaly. Minkowski referred to (28) and to the norm of a 4-velocity vector:

$$\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2 = \left(\frac{dt}{d\tau}\right)^2 - 1, \tag{29}$$

in order to determine the difference between the mean anomaly in coordinate time nt and the mean anomaly in proper time $n\tau$. From (29), Minkowski deduced:[107]

$$\left(\frac{dt}{d\tau}\right)^2 - 1 = \frac{m*}{ac^2}\,\frac{1 + e\cos E}{1 - e\cos E}. \tag{30}$$

Solving (30) for the coordinate time dt, expanding to terms in c^{-2}, and multiplying by n led Minkowski to the expression:

$$ndt = nd\tau\left(1 + \frac{1}{2}\,\frac{m*}{ac^2}\,\frac{1 + e\cos E}{1 - e\cos E}\right). \tag{31}$$

Recalling (28), Minkowski managed to express the difference between the mean anomaly in coordinate time and proper time:[108]

$$nt + \text{const.} = \left(1 + \frac{1}{2}\,\frac{m*}{ac^2}\right)n\tau + \frac{m*}{ac^2}\,e\sin E. \tag{32}$$

Evaluating the relativistic factor $m*/ac^2$ for solar mass and the Earth's semi-major axis to be 10^{-8}, Minkowski found the deviation from Newtonian orbits to be negligible in the solar system. On this basis, he concluded that

> a decision *against* such a law and the proposed modified mechanics in favor of the Newtonian law of attraction with Newtonian mechanics would not be deducible from astronomical observations.[109]

According to the quoted remark, there was more at stake here for Minkowski than just the empirical adequacy of his law of gravitational attraction, as his claim is for parity between Newton's law and classical mechanics, on one hand, and the *system* composed of the law of gravitation and spacetime mechanics on the other hand. This new system, Minkowski claimed, was verified by astronomical observations at least as well as the classical system formed by the Newtonian law of attraction and Newtonian mechanics.

Instead of comparing his law with one or the other of Poincaré's laws, Minkowski noted a difference in *method*, as mentioned above. In light of Minkowski's emphasis on the methodological difference with Poincaré, and the hybrid geometric-symbolic nature of Minkowski's exposition, it is clear that the point of reexamining the problem of relativity and gravitation in the *Grundgleichungen* was not simply to make plausible the inclusion of gravitation in a relativistic framework. Rather, since gravitational attraction was the only example Minkowski provided of his formalism in action, his line of argument served to *validate* his four-dimensional calculus, over and above the requirements of plausibility.

From the latter point of view, Minkowski had grounds for satisfaction, although one imagines that he would have preferred to find that his law diverged from Newton's law just enough to account for the observed anomalies. It stands to reason that if Minkowski had been fully satisfied with his first law, he would not have proposed a second law in his next paper–which turned out to be the last he would finish for publication. The latter article developed out of a well-known lecture entitled "Space and Time" (*Raum und Zeit*), delivered in Cologne on 21 September, 1908, to the mathematics section of the German Association of Scientists and Physicians in its annual meeting (Walter 1999a, 49).

In the final section of his Cologne lecture, Minkowski took up the Lorentz-Poincaré theory, and showed how to determine the field due to a point charge in arbitrary motion. On this occasion, just as in his earlier discussion of gravitation in the *Grundgleichungen*, Minkowski referred to a spacetime diagram, but this time he provided the diagram (Figure 3). Identifying the 4-vector potential components for the source charge on this diagram, Minkowski remarked that the Liénard-Wiechert law was a consequence of just these geometric relations.[110]

Minkowski then described the driving force between two point charges. Adopting dot notation for differentiation with respect to proper time, he wrote the driving force exerted on an electron of charge e_1 at point P_1 by an electron of charge e:

$$- ee_1 \left(\dot{t}_1 - \frac{\dot{x}_1}{c} \right) \mathfrak{K}, \tag{33}$$

where $\dot{t}_1$ and $\dot{x}_1$ are 4-velocity components of the test charge e_1 and $\mathfrak{K}$ is a certain

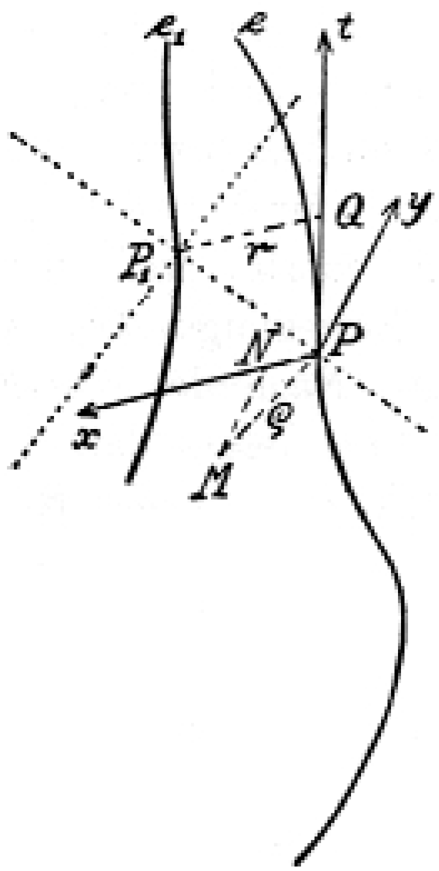

Figure 3: Minkowski's spacetime diagram of particle interaction (Minkowski 1909, 86).

4-vector. This was the first such description of the electrodynamic driving force due to a 4-vector potential, the simplicity of which, Minkowski claimed, compared favorably with the earlier formulations of Schwarzschild and Lorentz.[111]

In the same celebratory tone, Minkowski finished his article with a discussion of gravitational attraction. The "reformed mechanics", he claimed, dissolved the disturbing disharmonies between Newtonian mechanics and electrodynamics. In order to provide an example of this dissolution, he asked how the Newtonian law of attraction would sit with his principle of relativity. Minkowski continued:

> I will assume that if two point masses m, m_1 describe worldlines, a driving force vector is exerted by m on m_1, exactly like the one in the expression just given for the case of electrons, except that instead of $-ee_1$, we must now put in $+mm_1$.

Applying the substitution suggested by Minkowski to (33), we obtain:

$$mm_1 \left(\dot{t}_1 - \frac{\dot{x}_1}{c} \right) \mathfrak{K}, \tag{34}$$

where the coefficients m and m_1 refer to proper masses. Minkowski's new law of gravitation (34) fully expresses the driving force, unlike the formula (24) of his first law, which describes only one component. In addition, the 4-vectors are immediately identifiable from the notation alone. (In order to distinguish the law given in the *Grundgleichungen* from that of the Cologne lecture (34), I will call (34) Minkowski's second law.)

Since (33) was obtained from Lorentz-Poincaré theory via a 4-vector potential, the law of gravitation (34) ostensibly implied a 4-vector potential as well; in other words, following the example set by Poincaré's second law (10), Minkowski appealed in turn to a Maxwellian theory of gravitation similar to those of Heaviside, Lorentz, and Gans.[112] Although Minkowski made no effort to attach his law to these field theories, it was understood by Sommerfeld to be a formal consequence of just such a theory, as I will show in the next section.

What were the numerical consequences of this new law? Minkowski spared the reader the details, noting only that in the case of uniform motion of the source, the only divergence from a Keplerian orbit would stem from the replacement of coordinate time by proper time. He indicated that the numbers for this case had been worked out earlier, and his conclusion with respect to this new law was naturally the same: combined with the new mechanics, it was supported by astronomical observations to the same extent as the Newtonian law combined with classical mechanics.

Curiously enough, Minkowski offered no explanation of the need for a second law of attraction. Furthermore, by proposing two laws instead of one, Minkowski tacitly acknowledged defeat; despite his criticism of Poincaré's approach (see above, p. 19), he could hardly claim to have solved unambiguously the problem of gravitation. It may also seem strange that Minkowski discarded the differences between his new law (34) and the one he had proposed earlier.[113]

Minkowski revealed neither the motivation behind a second law of gravitation, nor why he neglected the differences between his two laws, but there is a straightforward way of explaining both of these mysteries. First, we recall the circumstances of his Cologne lecture, the final section of which Minkowski devoted to the theme of restoring unity to physics. What he wanted to stress on this occasion was that mechanics and electrodynamics harmonized in his four-dimensional scheme of things:

> In the mechanics reformed according to the world postulate, the disturbing disharmonies between Newtonian mechanics and modern electrodynamics fall out on their own.[114]

To support this view, Minkowski had to show that his reformed mechanics was a synthesis of classical mechanics and electrodynamics. A Maxwellian theory of gravitation fit the bill quite well, and consequently, Minkowski brought out his second law of gravitation (34). Clearly, this was not the time to point out the *differences* between his two laws. On the contrary, it was the perfect occasion to observe that a law of gravitation derived from a 4-vector potential formally identical to that of electrodynamics was observationally indistinguishable from Newton's law. Naturally, Minkowski seized this opportunity.

Sadly, Minkowski did not live long enough to develop his ideas on gravitation and electrodynamics; he died on 12 January, 1909, a few days after undergoing an operation for appendicitis. At the time, no objections to a field theory of gravitation analogous to Maxwell's electromagnetic theory were known, apart from Maxwell's own sticking-points(see above, p. 6).However, additional objections to this approach were raised by Max Abraham in 1912, after which the Maxwellian approach withered on the vine, as Gustav Mie and others pursued unified theories of electromagnetism and gravitation.[115]

Minkowski's first law of gravitation fared no better than his second law, but the four-dimensional language in which his two laws were couched had a bright future. The first one to use Minkowski's formal ideas to advantage was Sommerfeld, as we will see next.

3 Arnold Sommerfeld's hyper-Minkowskian laws of gravitation

Neither Poincaré's nor Minkowski's work on gravitation and relativity drew comment until 25 October, 1910, when the second installment of Arnold Sommerfeld's vectorial version of Minkowski's calculus, entitled "Four-dimensional vector analysis" *(Vierdimensionale Vektoranalysis)*, appeared in the *Annalen der Physik* (Sommerfeld 1910b). Sommerfeld's contribution differs from those of Poincaré and Minkowski in that it is openly concerned with the presentation of a new formalism, much as its title indicates. In this section, I discuss Sommerfeld's interest in vectors, the salient aspects of his 4-vector formalism, and his portrayal of Poincaré's and Minkowski's laws of gravitation.

Sommerfeld displayed a lively interest in vectors, beginning with his editorship of the physics volume of Klein's six-volume *Encyklopädie* in the summer of 1898.[116] He imposed a certain style of vector notation on his contributing authors, including typeface, terminology, symbolic representation of operations, units and dimensions, and the choice of symbols for physical quantities. Articles 12 to 14 of the physics volume appeared in 1904, and were the first to implement the notation scheme backed by Sommerfeld, laid out the same year in the *Physikalische Zeitschrift*.[117] While Sommerfeld belonged to the Vector Commission formed at Felix Klein's behest in 1902, it was clear to him as early as 1901 that the article on Maxwell's theory (commissioned to Lorentz) would serve as a "general directive" for future work in electrodynamics.[118] His intuition turned out to be correct: the principal "vector" of influence was Lorentz's Article 13 (Lorentz 1904b), featuring sections on vector notation and algebra, which set a *de facto* standard for vector approaches to electrodynamics.

As mentioned above(p. 16), only one effort to extend Poincaré's four-dimensional approach beyond the domain of gravitation was published prior to Minkowski's *Grundgleichungen*. By 1910, the outlook for relativity theory had changed due to the authoritative support of Planck and Sommerfeld, the announcement of experimental results favoring Lorentz's electron theory, and the broad diffusion (in 1909) of Minkowski's Cologne lecture. Dozens of physicists and mathematicians began to take an interest in relativity, resulting in a leap in relativist publications.[119]

The principal promoter of Minkowskian relativity, Sommerfeld must have felt by 1910 that it was the right moment to introduce a four-dimensional formalism. He was not alone in feeling this way, for three other formal approaches based on Minkowski's work appeared in 1910. Two of these were 4-vector systems, similar in some respects to Sommerfeld's, and worked out by Max Abraham and the American physical chemist Gilbert Newton Lewis, respectively. A third, non-vectorial approach was proposed by the Zagreb mathematician Vladimir Varičak. Varičak's was a real, four-dimensional, coordinate-based approach relying on hyperbolic geometry. Sommerfeld probably viewed this system as a potential rival to his own approach; although he did not mention Varičak, he wrote that a non-Euclidean approach was possible but could not be recommended (Sommerfeld 1910a, 752, note 1). Of the three alternatives to Sommerfeld's system, the non-Euclidean style pursued by Varičak and others was the only one to obtain even a modest following. An investigation of the reasons for the contemporary neglect of these alternative four-dimensional approaches is beyond the purview of our study; for what concerns us directly, none of these methods was applied to the problem of gravitation.[120]

Sommerfeld's paper, like those of Abraham, Lewis, and Varičak, emphasized formalism, and in this it differed from the *Grundgleichungen*, as mentioned above. Like the latter work, it focused attention on the problem of gravitation. Following the example set by both Poincaré and Minkowski, Sommerfeld capped his two-part *Annalen* paper with an application to gravitational attraction, which consisted of a reformulation, comparison and commentary of their work in his own terms. Not only was Sommerfeld's comparison of Poincaré's and Minkowski's laws of gravitation the first of its kind, it also proved to be the definitive analysis for his generation.

Sommerfeld's four-dimensional vector algebra and analysis offered no new 4-vector or 6-vector definitions, but it introduced a suite of 4-vector functions, notation, and vocabulary. The most far-reaching modification with respect to Minkowski's calculus was the elimination of lor (above, p. 26), in favor of extended versions of ordinary vector functions. In Sommerfeld's notational scheme, the ordinary vector functions div, rot, and grad (used by Lorentz in his *Encyklopädie* article on Maxwell's theory) were replaced by 4-vector counterparts marked by a leading capital letter: Div, Rot, and Grad. These three functions were joined by a 4-vector divergence marked by German typeface, $\mathfrak{Div}$. Sommerfeld chose to retain $\Box$ (cf. note 20), while noting the equivalence to his 4-vector functions: $\Box =$ Div Grad. The principal advantage of the latter functions was that their meaning was familiar to physicists. In the same vein, Sommerfeld supplanted Minkowski's unwieldy terminology of "spacetime vectors of the first and second type" (*Raum-Zeit-Vektoren I^ter und II^ter Art*) with the more succinct "four-vector" (*Vierervektor*) and "six-vector" (*Sechservektor*). The result was a compact and transparent four-dimensional formalism differing as little as possible from the ordinary vector algebra employed in the physics volume of the *Encyklopädie*.[121]

To show how his formalism performed in action, Sommerfeld first took up the geometric interpretation and calculation of the electrodynamic 4-vector potential and 4-force. In the new notation, Sommerfeld wrote the electrodynamic 4-force $\mathfrak{K}$ between two point charges e and e_0 in terms of three components in the direction of the lightlike 4-vector $\mathfrak{R}$, the source 4-velocity $\mathfrak{B}$, and the 4-acceleration $\dot{\mathfrak{B}}$:

$$\begin{aligned}
4\pi\mathfrak{K}_{\mathfrak{R}} &= \frac{ee_0}{c(\mathfrak{R}\mathfrak{B})^2}\left(\frac{c^2-(\mathfrak{R}\dot{\mathfrak{B}})}{(\mathfrak{R}\mathfrak{B})}(\mathfrak{B}_0\mathfrak{B})+(\mathfrak{B}_0\dot{\mathfrak{B}})\right)\mathfrak{R},\\
4\pi\mathfrak{K}_{\mathfrak{B}} &= \frac{-ee_0}{c(\mathfrak{R}\mathfrak{B})^2}\frac{c^2-(\mathfrak{R}\dot{\mathfrak{B}})}{(\mathfrak{R}\mathfrak{B})}(\mathfrak{B}_0\mathfrak{R})\mathfrak{B},\\
4\pi\mathfrak{K}_{\dot{\mathfrak{B}}} &= \frac{-ee_0}{c(\mathfrak{R}\mathfrak{B})^2}\frac{c^2-(\mathfrak{R}\dot{\mathfrak{B}})}{(\mathfrak{R}\mathfrak{B})}(\mathfrak{B}_0\mathfrak{R})\dot{\mathfrak{B}},
\end{aligned} \tag{35}$$

where parentheses indicate scalar products. Sommerfeld was careful to note the equivalence between (35) and what he called Minkowski's "geometric rule" (33).

In the ninth and final section of his paper, Sommerfeld took up the law of electrostatics and the classical law of gravitation. The former was naturally considered to be a special case of (35), with two point charges relatively at rest. The same was true for the law of gravitation, as Sommerfeld noted that Minkowski had proposed a formal variant of (33) as a law of gravitational attraction (what I call Minkowski's second law, (34)). Sommerfeld's expression of the electrodynamic 4-force is unwieldy, but takes on a simpler form in case of uniform motion of the source ($\dot{\mathfrak{B}} = 0$). Neglecting the 4π factor, and substituting $-mm_0$ for $+ee_0$, Sommerfeld expressed the corresponding version of Minkowski's second law:

$$-mm_0c\frac{(\mathfrak{B}_0\mathfrak{B})\mathfrak{R}-(\mathfrak{B}_0\mathfrak{R})\mathfrak{B}}{(\mathfrak{R}\mathfrak{B})^3}. \tag{36}$$

The latter law is compact and self-contained, in that its interpretation depends only on the definitions and rules of the algebraic formalism. In this sense, (36) improves on the Minkowskian (34), even if it represents only a special case of the latter law.

Once Sommerfeld had expressed Minkowski's second law in his own terms, he turned to Poincaré's two laws. The transformation of Poincaré's first law was more laborious than the transformation of Minkowski's second law. First of all, Sommerfeld transcribed Poincaré's first law (9) into his 4-vector notation, while retaining the original designation of invariants. This step itself was not simple: in order to cast Poincaré's kinematic invariants as scalar products of 4-vectors, Sommerfeld had to adjust the leading sign of (9), to obtain:

$$\frac{k_0\mathfrak{K}}{mm'} = -\frac{1}{B^3C}\left(C\mathfrak{R}-\frac{1}{c}A\mathfrak{B}\right). \tag{37}$$

Sommerfeld noted the "correction" of what he called an "obvious sign error" in (9).[122] The difference is due to Poincaré's irregular derivation of the kinematic invariants (1), as mentioned above(p. 8), although from Sommerfeld's remark it is not clear that he saw it this way.

The transformation of Poincaré's second law (10) was less straightforward. It appears that instead of deriving a 4-vector expression as in the previous case, Sommerfeld followed Poincaré's lead by eliminating the Lorentz-invariant factor C from the denominator on the right-hand side of the first law (37), which results in the equation:

$$\frac{k_0\mathfrak{K}}{mm'} = -\frac{1}{B^3}\left(C\mathfrak{R} - \frac{1}{c}A\mathfrak{B}\right). \tag{38}$$

Sommerfeld expressed Poincaré's kinematic invariants A, B, and C as scalar products:

$$A = -\frac{1}{c}(\mathfrak{R}\mathfrak{B}_0), \qquad B = -\frac{1}{c}(\mathfrak{R}\mathfrak{B}), \qquad C = -\frac{1}{c^2}(\mathfrak{B}_0\mathfrak{B}). \tag{39}$$

He also replaced the mass term m' in (37) and (38) by the product of rest mass m_0 and the Lorentz factor k_0, i.e., $m' = m_0 k_0$. At this point, he could express Poincaré's two laws exclusively in terms of constants, scalars, and 4-vectors:

$$mm_0c^3\frac{(\mathfrak{B}_0\mathfrak{B})\mathfrak{R} - (\mathfrak{B}_0\mathfrak{R})\mathfrak{B}}{(\mathfrak{R}\mathfrak{B})^3(\mathfrak{B}_0\mathfrak{B})}, \tag{40}$$

$$-mm_0c\frac{(\mathfrak{B}_0\mathfrak{B})\mathfrak{R} - (\mathfrak{B}_0\mathfrak{R})\mathfrak{B}}{(\mathfrak{R}\mathfrak{B})^3} \tag{41}$$

In the latter form, Sommerfeld's (approximate) version of Minkowski's second law (36) matches exactly his (exact) version of Poincaré's second law (41). Sommerfeld pointed out this equivalence, and noted again that the difference between (40) and (41) amounted to a single factor, in the scalar product of 4-velocities: $C = -(\mathfrak{B}_0\mathfrak{B})/c^2$. (All six Lorentz-invariant laws of gravitation of Poincaré, Minkowski, and Sommerfeld are presented in Table 1.) Sommerfeld summed up his result by saying that when the acceleration of the active mass is neglected, Minkowski's special formulation of Newton's law (34) is subsumed by Poincaré's indeterminate formulation. In other words, the approximate form of Minkowski's second law was captured by Poincaré's remark that his first law (9) could be multiplied by an unlimited number of Lorentz-invariant quantities (within certain constraints).

The message of the basic equivalence of Poincaré's pair of laws to Minkowski's pair echoes the latter's argument in his Cologne lecture, to the effect that spacetime mechanics removed the disharmonies of classical mechanics and electrodynamics (see above, p. 36).This message was reinforced by Sommerfeld's graphical representation of the 4-vector components of these laws in a spacetime diagram,

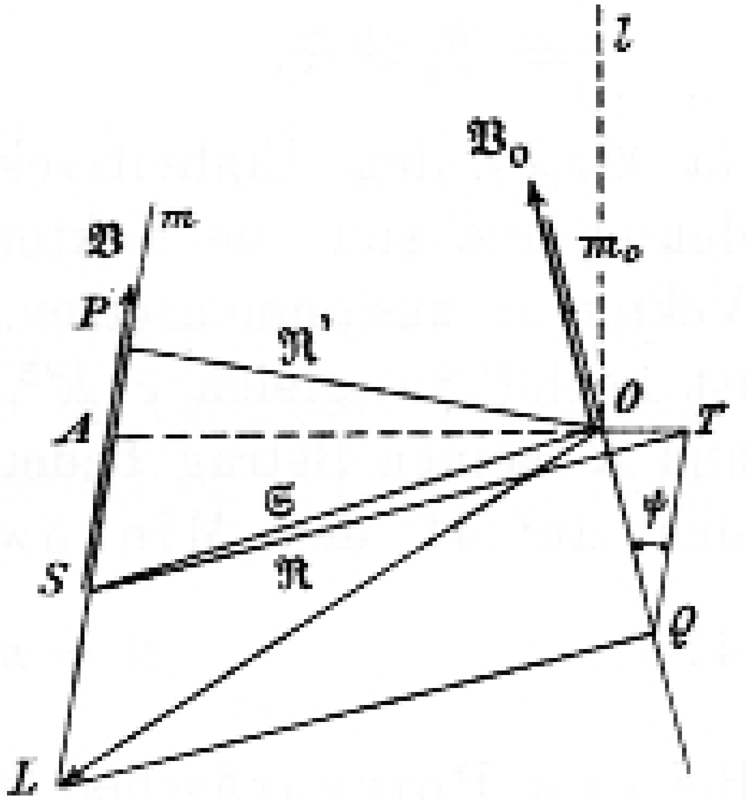

Figure 4: Sommerfeld's illustration of the two laws of gravitation (Sommerfeld 1910b, 687).

reproduced in Figure 4. The 4-vector relations in (40) and (41) are shown in the figure; the worldline of the active mass m appears on the left-hand side of the diagram, and the line OL (which coincides with $\mathfrak{R}$) lies on the retrograde lightcone from the origin O on the worldline of the passive mass m_0. All three 4-vectors in (40) and (41), $\mathfrak{R}$, $\mathfrak{B}_0$, and $\mathfrak{B}_0$ are represented in the diagram, along with an angle ψ corresponding to the Lorentz-invariant $C = \cos\psi$ distinguishing (40) and (41).[123]

So far, Sommerfeld had dealt with three of the four laws of gravitation, leaving out only Minkowski's first law. Since Minkowski's presentation of his first law was a purely geometric affair, Sommerfeld had no choice but to reconstruct his argument with reference to a spacetime diagram describing the components of (24) in terms of the angle ψ and a fourth 4-vector, $\mathfrak{S}$. He showed the numerator in (40) and (41) to be equal to the product $(\mathfrak{B}_0\mathfrak{B})\mathfrak{S}$, and expressed the denominator of (41) in terms of the length R' of the 4-vector $\mathfrak{R}'$ in Figure 4, to obtain the

formula:
$$\mathfrak{K} = mm_0 \cos\psi \frac{\mathfrak{S}}{R'^3}, \tag{42}$$
which he showed to be equivalent to (40). Eliminating the factor $C = \cos\psi$ from the latter equation, Sommerfeld obtained an expression for (41) in terms of $\mathfrak{S}$:
$$\mathfrak{K} = \frac{mm_0\mathfrak{S}}{R'^3}. \tag{43}$$
The latter two driving force equations, (42) and (43), were thus rendered geometrically by Sommerfeld, facilitating the comprehension of their respective vector-symbolic expressions (40) and (41).

In general, the driving force of (42) is weaker, *ceteris paribus,* than that of (43) due to the cosine in the former, but Sommerfeld did not develop these results numerically, noting only that the four laws were equally valid from an empirical standpoint.[124] He noted that Poincaré's analysis allowed for several other laws, but that in all cases, one sticking-point remained: there was no answer to the question of how to localize momentum in the gravitational field.

By rewriting Poincaré's and Minkowski's laws in his new 4-vector formalism, Sommerfeld effectively rationalized their contributions for physicists. The goal of his paper, announced at the outset, was to display the "remarkable simplification of electrodynamic concepts and calculations" resulting from "Minkowski's profound spacetime conception."[125] Actually, Sommerfeld's comparison of Poincaré's and Minkowski's laws of gravitation was designed to show *his* formalism in an attractive light. In realizing this comparison in his own formalism, Sommerfeld smoothed out the idiosyncrasies of Poincaré's method, inappropriately lending him a 4-vector approach. He felt that Poincaré had "already employed 4-vectors" (Sommerfeld 1910b, 685), although as shown in the first section, Poincaré's use of four-dimensional entities was tightly circumscribed by the objective of formulating Lorentz-invariants. In Thomas Kuhn's optical metaphor (Kuhn 1970, 112), Sommerfeld read Poincaré's theory through a Minkowskian lens; in other words, he read it as a spacetime theory. For Sommerfeld, no less than for Minkowski, the discussion of gravitation and relativity was modulated by the programmatic objective of promoting a four-dimensional formalism. Satisfying this objective without ignoring Poincaré's work, however, meant rationalizing Poincaré's contribution.[126]

Sommerfeld's reading of Minkowski's second law contrasts with its muted exposition in the original text above(p. 36), in that he gave it pride of place with respect to the other three laws. This change in emphasis on Sommerfeld's part reflects his own research interests in electrodynamics, and his outlook on the future direction of physics.[127] But what originally motivated him to propose a 4-dimensional formalism? The inevitability of a 4-dimensional vector algebra as a standard tool of the physicist was probably a foregone conclusion for him by 1910, such that the promotion of the ordinary vector notation used in the *Encyklopädie* obliged him to propose essentially the same notation for 4-vectors. Sommerfeld referred modestly to his work as an "explanation of Minkowskian ideas" (Sommerfeld 1910a, 749), but as he explained to his friend Willy Wien, co-editor with Planck of the *Annalen der Physik*, Minkowski's original 4-vector scheme had evolved. "The geometrical systematics" Sommerfeld announced, "is now hyper-Minkowskian."[128] In the same letter to Wien, Sommerfeld confessed that his paper had required substantial effort, and he expressed doubt that it would prove worthwhile. Sommerfeld displayed either pessimism or modesty here, but in fact his effort was richly rewarded, as his streamlined four-dimensional algebra and analysis quickly won both Einstein's praise and the confidence of his contemporaries.[129]

Sommerfeld's work was eagerly read by young theoretical physicists raised in the heady atmosphere of German vectorial electrodynamics. One of the early adepts of Sommerfeld's formalism was Philipp Frank (1884–1966), who was then a Privatdozent in Vienna. By way of introduction to his 1911 study of the Lorentz-covariance of Maxwell's equations, Frank described the new four-dimensional algebra as a combination of "Sommerfeld's intuitiveness with Minkowski's mathematical elegance" (Frank 1911, 600). He recognized, however, that of late, physicists had been overloaded with outlandish symbolic systems and terminology, and promised to stay within the boundaries of Sommerfeld's system, at least as far as this was possible.

Physicists were indeed inundated in 1910–1911 with a bewildering array of new symbolic systems, including an ordinary vector algebra (Burali-Forti & Marcolongo 1910), and a quaternionic calculus (Conway 1911), in addition to the hyperbolic-coordinate system and three 4-vector formalisms already mentioned. By 1911, 4-vector and 6-vector operations featured prominently in the pages of the *Annalen der Physik*. Out of the nine theoretical papers concerning relativity theory published in the *Annalen* that year, five made use of a four-dimensional approach to physics, either in terms of 4-vector operations, or by referring to spacetime coordinates. Four out of five authors of "four-dimensional" papers cited Minkowski's or Sommerfeld's work; the fifth referred to Max Laue's new relativity textbook (Laue 1911). This timely and well-written little book went far in standardizing the terminology and notation of four-dimensional algebra, such that by January of 1912, Max Abraham preferred the Sommerfeld-Laue notation to his own for the exposition of his theory of gravitation (Abraham 1910, 1912a, 1912b).

While young theorists were quick to pick up on the Sommerfeld-Laue calculus, textbook writers did not follow the trend. Of the four textbooks to appear on relativity in 1913–1914, only the second edition of Laue's book (Laue 1913) employed this formalism. Ebenezer Cunningham presented a 4-dimensional approach based on Minkowski's work, but explicitly rejected Sommerfeld's "quasi-geometrical language", which conflicted with his own purely algebraic presentation (Cunningham 1914, 99). A third textbook by Ludwik Silberstein (1914), a former student of Planck, gave preference to a quaternionic presentation, while the fourth, by Max B. Weinstein (1913), opted for Cartesian coordinates. Curiously enough, Weinstein dedicated his work to the memory of Minkowski. Apparently disturbed by this profession of fidelity, Max Born, who had briefly served as Minkowski's assistant, deplored the form of Weinstein's approach to relativity:

> [Minkowski] put perhaps just as much value on his presentation as on its content. For this reason, I do not believe that entrance to his conceptual world is facilitated when it is overwhelmed by an enormous surfeit of formulas.[130]

By this time, Born himself had dropped Minkowski's formalism in favor of the Sommerfeld-Laue approach, such that the target of his criticism was Weinstein's disregard for 4-dimensional methods in general, and not the neglect of Minkowski's matrix calculus.[131] What Born was pointing out here was that it had become highly impractical to study the theory of relativity without recourse to a 4-dimensional formalism. This may explain why Laue's was the only one of the four textbooks on relativity to be reedited, reaching a sixth edition in 1955.

In summary, the language developed by Sommerfeld for the expression of the laws of gravitation of Poincaré and Minkowski endured, while the laws themselves remained tentative at best. This much was clear as early as 1912, when Jun Ishiwara reported from Japan on the state of relativity theory. This theory, Ishiwara felt, had shed no light on the problem of gravitation, with a single exception: Minkowski and Sommerfeld's "formal mathematical treatment" (Ishiwara 1912, 588). The trend from Poincaré to Sommerfeld was one of increasing reliance on formal techniques catering to Lorentz-invariance; in the space of five years, the physical content of the laws of gravitation remained stable, while their formal garb evolved from Cartesian to hyper-Minkowskian.

[a]Mass terms are neglected, such that the right-hand side of each equation is implicitly multiplied by the product of the two masses. When both sides of the four equations are multiplied by the factor k_0, they express components of a 4-vector, $k_0(X_1, Y_1, Z_1, iT_1)$. The constants k_0 and k_1 are defined as: $k_0 = 1/\sqrt{1-\sum \xi^2}$ and $k_1 = 1/\sqrt{1-\sum \xi_1^2}$. A, B, and C denote the last three Lorentz-invariants in (1): $A = \frac{t-\sum x\xi}{\sqrt{1-\sum \xi^2}}$, $B = \frac{t-\sum x\xi_1}{\sqrt{1-\sum \xi_1^2}}$, $C = \frac{1-\sum \xi\xi_1}{\sqrt{(1-\sum \xi^2)(1-\sum \xi_1^2)}}$, where $\sum \xi$ and $\sum \xi_1$ designate the ordinary velocities of the passive and active mass points, with components ξ, η, ζ, and ξ_1, η_1, ζ_1. The time t is set equal to the negative distance between the passive mass point and the retarded position of the active mass point, $t = -\sqrt{\sum x^2} = -r$. Poincaré's second law is shown in the bottom

Table 1: Lorentz-invariant laws of gravitation, 1906–1910

Poincaré (1906)[a]	**Minkowski (1908)**[b]	**Sommerfeld (1910)**[c]
$X_1 = \frac{x}{k_0 B^3} - \xi_1 \frac{k_1}{k_0} \frac{A}{B^3 C}$ $Y_1 = \frac{y}{k_0 B^3} - \eta_1 \frac{k_1}{k_0} \frac{A}{B^3 C}$ $Z_1 = \frac{z}{k_0 B^3} - \zeta_1 \frac{k_1}{k_0} \frac{A}{B^3 C}$ $T_1 = -\frac{r}{k_0 B^3} - \frac{k_1}{k_0} \frac{A}{B^3 C}$	$mm^* \left(\frac{OA'}{B^* D^*}\right)^3 BD^*$	$mm_0 c^3 \frac{(\mathfrak{B}_0 \mathfrak{B})\mathfrak{R} - (\mathfrak{B}_0 \mathfrak{R})\mathfrak{B}}{(\mathfrak{R}\mathfrak{B})^3 (\mathfrak{B}_0 \mathfrak{B})}$
$X_1 = \frac{\lambda}{B^3} - \frac{\eta\nu' - \zeta\mu'}{B^3}$ $Y_1 = \frac{\mu}{B^3} - \frac{\zeta\lambda' - \xi\nu'}{B^3}$ $Z_1 = \frac{\nu}{B^3} - \frac{\xi\mu' - \eta\lambda'}{B^3}$	$mm_1 \left(\dot{t}_1 - \frac{\dot{x}_1}{c}\right) \mathfrak{K}$	$-mm_0 c \frac{(\mathfrak{B}_0 \mathfrak{B})\mathfrak{R} - (\mathfrak{B}_0 \mathfrak{R})\mathfrak{B}}{(\mathfrak{R}\mathfrak{B})^3}$

row; he neglected to write the fourth component T_1, determined from the first three by the orthogonality condition $T_1 = \sum X_1 \xi$. The new variables in the bottom row are:

$$\lambda = k_1(x + r\xi_1), \qquad \mu = k_1(y + r\eta_1), \qquad \nu = k_1(z + r\zeta_1),$$
$$\lambda' = k_1(\eta_1 z - \zeta_1 y), \quad \mu' = k_1(\zeta_1 x - \xi_1 z), \quad \nu' = k_1(\xi_1 y - x\eta_1).$$

[b]The formula in the top row describes the first three components of the driving force; the fourth component is obtained analytically. The constants m and m* designate the passive and active proper mass, respectively, while the remaining letters stand for space-time points, as reconstructed in Figure 1(p. 29).The formula in the bottom row represents the driving force of gravitation as described, but not formally expressed, by Minkowski (1909). The constants m and m_1 designate the active and passive proper mass, $\dot{t}_1$ and $\dot{x}_1$ are 4-velocity components of the passive mass, c is the speed of light and $\mathfrak{K}$ is a 4-vector, for the definition of which see note 111.

[c]The constants m_0 and m designate the passive and active proper mass, respectively, c denotes the speed of light, $\mathfrak{B}_0$ and $\mathfrak{B}$ represent the corresponding 4-velocities, and $\mathfrak{R}$ stands for the lightlike interval between the mass points.

4 Conclusion: On the emergence of the four-dimensional view

After a century-long process of accommodation to the use of tensor calculus and spacetime diagrams for analysis of physical interactions, the mathematical difficulties encountered by the pioneers of 4-dimensional physics are hard to come to terms with. Not only is the oft-encountered image of flat-spacetime physics as a trivial consequence of Einstein's special theory of relativity and Felix Klein's geometry consistent with such accommodation, it reflects Minkowski's own characterization of the background of the four-dimensional approach(p. 22). However, this description ought not be taken at face value, being better understood as a rhetorical ploy designed to induce mathematicians to enter the nascent field of relativistic physics (Walter 1999a). When the principle of relativity was formulated in 1905, even for one as adept as Henri Poincaré in the application of group methods, the path to a four-dimensional language for physics appeared strewn with obstacles. Much as Poincaré had predicted (above, p. 15), the construction of this language cost Minkowski and Sommerfeld considerable pain and effort.

Clear-sighted as he proved to be in this regard, Poincaré did not foresee the emergence of forces that would accelerate the construction and acquisition of a four-dimensional language. With hindsight, we can identify five factors favoring the use and development of a four-dimensional language for physics between 1905 and 1910: the elaboration of new concepts and definitions, the introduction of a graphic model of spacetime, the experimental confirmation of relativity theory, the vector-symbolic movement, and problem-solving performance.

In the beginning, the availability of workable four-dimensional concepts and definitions regulated the analytic reach of a four-dimensional approach to physics. Poincaré's discovery of the 4-vectors of velocity and force in the course of his elaboration of Lorentz-invariant quantities, and Minkowski's initial misreading of Poincaré's definitions underline how unintuitive these notions appeared to turn-of-the-century mathematicians. The lack of a 4-velocity definition visibly hindered Minkowski's elaboration of spacetime mechanics and theory of gravitation. It is remarkable that even after Minkowski presented the notions of proper time, worldline, rest-mass density, and the energy-momentum tensor, putting the spacetime electrodynamics and mechanics on the same four-dimensional footing, his approach failed to convince physicists. Nevertheless, all of these discoveries extended the reach of the four-dimensional approach, in the end making it a viable candidate for the theorist's toolbox.

Next, Minkowski's visually-intuitive spacetime diagram played a decisive role in the emergence of the four-dimensional view. While the spacetime diagram reflects some of the concepts mentioned above, its utility as a cognitive tool exceeded by far that of the sum of its parts. In Minkowski's hands, the spacetime diagram was more than a tool, it was a model used to present both of his laws of gravitation. Beyond their practical function in problem-solving, spacetime diagrams favored the diffusion in wider circles of both the theory of relativity and the four-dimensional view of this theory, in particular among non-mathematicians, by providing a visually intuitive means of grasping certain consequences of the theory of relativity, such as time dilation and Lorentz contraction. Minkowski's graphic model of spacetime thus enhanced both formal and intuitive approaches to special relativity.

In the third place, the ultimate success of the four-dimensional view hinged on the empirical adequacy of the theory of relativity. It is remarkable that the conceptual groundwork, and much of the formal elaboration of the four-dimensional view was accomplished during a time when the theory of relativity was less well corroborated by experiment than its rivals. The reversal of this situation in favor of relativity theory in late 1908 favored the reception of the existing four-dimensional methods, and provided new impetus both for their application and extension, and for the development of alternatives, such as that of Sommerfeld.

The fourth major factor influencing the elaboration of a four-dimensional view of physics was the vector-symbolic movement in physics and mathematics at the turn of the twentieth century (McCormmach 1976, xxxi). The participants in this movement, in which Sommerfeld was a leading figure, believed in the efficacy of vector-symbolic methods in physics and geometry, and sought to unify the plethora of notations employed by various writers. The movement's strength varied from country to country; it was largely ignored in France, for example, in favor of the coordinate-based notation favored by Poincaré and others. Poincaré's pronounced disinterest in the application and development of a four-dimensional calculus for physics was typical of contemporary French attitudes toward vector-symbolic methods. In Germany, on the other hand, electrodynamicists learned Maxwell's theory from the mid-1890s in terms of curl $\mathfrak{h}$ and div $\mathfrak{E}$. In Zürich and Göttingen during this period, Minkowski instructed students – including Einstein – in the ways of the vector calculus. Unlike Poincaré, Minkowski was convinced that a four-dimensional language for physics would be worth the effort spent on its elaboration, yet he ultimately abandoned the vector-symbolic model in favor of an elegant and sophisticated matrix calculus. This choice was deplored by physicists (including Einstein), and mooted by Sommerfeld's conservative extension of the standard vector formalism into an immediately successful 4-vector algebra and analysis. In sum, the vector-symbolic movement functioned alternatively as an accelerator of the elaboration of four-dimensional calculi (existing systems served as templates), and as a regulator (penalizing Minkowski's neglect of standard vector operations).

The fifth and final parameter affecting the emergence of the four-dimensional view of physics was problem-solving performance. From the standpoint of ease of calculation, any four-dimensional vector formalism at all compared well to a Cartesian-coordinate approach, as Weinstein's textbook demonstrated; the advantage of ordinary vector methods over Cartesian coordinates was less pronounced. As we have seen, Poincaré applied his approach to the problem of constructing a Lorentz-invariant law of gravitational attraction, and was followed in turn by Minkowski and Sommerfeld, both of whom also provided examples of problem-solving. In virtue of the clarity and order of Sommerfeld's detailed, coordinate-free comparison of the laws of gravitation of Poincaré and Minkowski, his 4-vector algebra appeared to be the superior four-dimensional approach, just when physicists and mathematicians were turning to relativity in greater numbers.

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Notes

[1]In limiting the scope of this paper to the methods applied by their authors to the problem of gravitation, four contributions to four-dimensional physics are neglected: that of Richard Hargreaves, based on integral invariants (1908), two 4-vector systems due to Max Abraham (1910) and Gilbert Newton Lewis (1910a), and Vladimir Varičak's hyperbolic-function based approach (1910).

[2]For an overview of research on gravitation from 1850 to 1915, see Roseveare (1982). On early 20th-century investigations of gravitational absorption, see Andrade Martins (1999). While only Lorentz-covariant theories are considered in this paper, the relative acceptance of the principle of relativity among theorists is understood as one parameter among several influencing the development of four-dimensional physics.

[3]Buchwald (1985, 242), Darrigol (2000, 325), Buchwald (2001).

[4]Lorentz took the force per unit charge on a volume element of charged matter moving with velocity $\mathfrak{v}$ in the electric and magnetic fields $\mathfrak{d}$ and $\mathfrak{h}$ to be $\mathfrak{f} = \mathfrak{d} + \frac{1}{c}[\mathfrak{v} \cdot \mathfrak{h}]$, where the brackets indi-

cate a vector product (Lorentz 1904c, 156–157). For a comparison of electrodynamic Lagrangians from Maxwell to Schwarzschild, see Darrigol (2000, App. 9).

[5]On the Maxwellian approach to gravitation, see North (1965, chap. 3), Roseveare (1982, 129–31), and Norton (1992, 32). The distinction drawn here between retarded action at a distance and field representations reflects that of Lorentz (1904b), for whom this was largely a matter of convenience. On nineteenth-century conceptions of the electromagnetic field, see Cantor & Hodge (1981).

[6]On Poincaré's theory see Cunningham (1914, 173), Whitrow (1965, 20), Harvey (1965, 452), Cuvaj (1970, App. 5), Schwartz (1972), Zahar (1989, 192), Torretti (1996, 132). On Minkowski's theory see Weinstein (1914, 61), Pyenson (1985a, 88), Corry (1997, 287).

[7]Laplace estimated the propagation velocity of gravitation to be 10^6 times that of light, and Poincaré noted that such a signal velocity would allow inertial observers to detect their motion with respect to the ether (Poincaré 1904, 312).

[8]Poincaré (1906, 153–154); Miller (1973, 230–233). Following Abraham's account (Abraham 1905, 205), the problem may be presented in outline as follows (using modified notation and units). Consider a deformable massless sphere of radius a and uniformly distributed surface charge, and assume that this is a good model of the electron. The longitudinal mass $m_{\shortparallel}$ of this sphere may be defined as the quotient of external force and acceleration, $m_{\shortparallel} = d|\mathbf{G}|/d|\mathbf{v}|$, where $\mathbf{G}$ is the electromagnetic momentum resulting from the electron's self-fields, and $\mathbf{v}$ is electron velocity. Defining the electromagnetic momentum to be $\mathbf{G} = \int \mathbf{E} \times \mathbf{B} dV$, where $\mathbf{E}$ and $\mathbf{B}$ denote the electric and magnetic self-fields, and V is for volume, we let $c = 1$, and find the longitudinal mass for small velocities to be $m_{\shortparallel} = \frac{e^2}{6\pi a}\left(1 - v^2\right)^{-3/2}$. Longitudinal electron mass may also be defined in terms of the electromagnetic energy W of the electron's self-fields, assuming quasistationary motion: $m_{\shortparallel} = \frac{1}{|\mathbf{v}|}\frac{dW}{d|\mathbf{v}|}$, where $W = \frac{e^2}{6\pi a}\left(1 - v^2\right)^{-1/2} + \frac{e^2}{24\pi a}\left(1 - v^2\right)^{1/2}$. This leads, however, to an expression for longitudinal mass different from the previous one: $m_{\shortparallel} = \frac{e^2}{6\pi a}\left[\left(1 - v^2\right)^{-3/2} + \frac{1}{4}\left(1 - v^2\right)^{-1/2}\right]$. From the difference in these two expressions for longitudinal mass, Abraham concluded that the Lorentz electron required the postulation of a non-electromagnetic force and was thereby not compatible with a purely electromagnetic foundation of physics.

[9]See Poincaré (1885, 1902a, 1902b). In the limit of null angular velocity, gravitational attraction can be replaced by electrostatic repulsion, with a sign reversal in the pressure gradient.

[10]Einstein (1905, 917). Poincaré also neglected the mass contribution of the binding potential in his 1906–1907 Sorbonne lectures, according to student notes (Poincaré 1953, 233). For reviews of Poincaré's derivation of the binding potential, see Cuvaj (1970, App. 11) and Miller (1973). On post-Minkowskian interpretations of the binding potential (also known as Poincaré pressure), see Cuvaj (1970, 203), Miller (1981, 382, n. 29), and Yaghjian (1992).

[11]In this paper Poincaré made no distinction between inertial and gravitational mass.

[12]As Cuvaj (1968, 1112) points out, Poincaré may have found inspiration for this conjecture in Paul Langevin's remark that gravitation stabilized the electron against Coulomb repulsion. Unlike Langevin, Poincaré anticipated a unified theory of gravitation and electrons, in the spirit of theories pursued later by Gustav Mie, Gunnar Nordström, David Hilbert, Hans Reissner, Hermann Weyl and Einstein; for an overview see Vizgin (1994).

[13]"Ainsi la théorie de Lorentz expliquerait complètement l'impossibilité de mettre en évidence le mouvement absolu, si toutes les forces étaient d'origine électromagnétique" (Poincaré 1906, 166).

[14]See Lorentz (1900), Havas (1979, 83), Torretti (1996, 131). On Lorentz's precursors see Whittaker (1951, 2:149) and Zenneck (1903). Lorentz's theory of gravitation failed to convince Oliver Heaviside, who had carefully weighed the analogy from electromagnetism to gravitation

(1893). In a letter to Lorentz, Heaviside called into question the theory's electromagnetic nature, by characterizing Lorentz's gravitational force as "action at a distance of a double kind" (18 July, 1901, Lorentz Papers, Rijksarchief in Noord-Holland te Haarlem). Aware of these difficulties, Lorentz eventually discarded his theory, citing its incompatibility with the principle of relativity (Lorentz 1914, 32).

[15]In his 1906–1907 Sorbonne lectures (1953), Poincaré discussed a different theory (based on an idea due to Le Sage) that Lorentz had proposed in the same paper, without mentioning the Mossotti-style theory. His first discussion of the latter theory was in 1908, when he considered it to be an authentic relativistic theory, and one in which the force of gravitation was of electromagnetic origin (Poincaré 1908, 399).

[16]Poincaré (1906, 166). Poincaré's account of Lorentz's reasoning should be taken with a grain of salt, as Lorentz made no mention of his theory of gravitation in the 1904 publication referred to by Poincaré, "Electromagnetic phenomena in a system moving with any velocity less than that of light." While the electron theory developed in the latter paper did not address the question of the origin of the gravitational force, it admitted the possibility of a reduction to electromagnetism (such as that of his own theory) by means of the additional hypothesis referred to in the quotation: all forces of interaction transformed in the same way as electric forces in an electrostatic system (Lorentz 1904a, § 8). The contraction hypothesis formerly invoked to account for the null result of the Michelson-Morley experiments, Lorentz added, was subsumed by the new hypothesis.

[17]"Regardons x, y, z, $t\sqrt{-1}$, δx, δy, δz, $\delta t\sqrt{-1}$, $\delta_1 x$, $\delta_1 y$, $\delta_1 z$, $\delta_1 t\sqrt{-1}$, comme les coordonnées de 3 points P, P', P'' dans l'espace à 4 dimensions. Nous voyons que la transformation de Lorentz n'est qu'une rotation de cet espace autour de l'origine, regardée comme fixe. Nous n'aurons donc pas d'autres invariants distincts que les six distances des trois points P, P', P'' entre eux et à l'origine, ou, si l'on aime mieux, que les 2 expressions : $x^2 + y^2 + z^2 - t^2$, $x\delta x + y\delta y + z\delta z - t\delta t$, ou les 4 expressions de même forme qu'on en déduit en permutant d'une manière quelconque les 3 points P, P', P''" (Poincaré 1906, 168–169).

[18]Poincaré's three points P, P', P'' may be interpreted in modern terminology as follows. Let the spacetime coordinates of the passive mass point be $A = (x_0, y_0, z_0, t_0)$, with ordinary velocity $\xi = (\delta x/\delta t, \delta y/\delta t, \delta z/\delta t)$, such that at time $t_0 + \delta t$ it occupies the spacetime point $A' = (x_0 + \delta x, y_0 + \delta y, z_0 + \delta z, t_0 + \delta t)$. Likewise for the active mass point, $B = (x_0 + x, y_0 + y, z_0 + z, t_0 + t)$, with ordinary velocity $\xi_1 = (\delta_1 x/\delta_1 t, \delta_1 y/\delta_1 t, \delta_1 z/\delta_1 t)$, such that at time $t_0 + t + \delta_1 t$, it occupies the spacetime point $B' = (x_0 + x + \delta_1 x, y_0 + y + \delta_1 y, z_0 + z + \delta_1 z, t_0 + t + \delta_1 t)$. Poincaré's three quadruplets may now be expressed as position 4-vectors: $P = B - A$, $P' = B' - B$, $P'' = A' - A$.

[19]While the first German textbook on electromagnetism to employ vector notation systematically dates from 1894 (Föppl 1894), the first comparable textbook in French was published two decades later by Jean-Baptiste Pomey (1861–1943), instructor of theoretical electricity at the *École supérieure des Postes et Télégraphes* in Paris (Pomey 1914).

[20]The Laplacian was expressed generally as $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$, but by Poincaré as Δ. The d'Alembertian, $\Box \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2 - \partial^2/\partial t^2$, became in Poincaré's notation: $\Box \equiv \Delta - d^2/dt^2$. Poincaré employed $\Box$ in his lectures on electricity and optics (Poincaré 1901, 456), and was the first to employ it in a relativistic context.

[21]Poincaré's manuscript lecture notes for celestial mechanics, however, show that he saw fit to introduce the quaternionic method to his students (undated notebook on quaternions and celestial mechanics, 32 pp., private collection, Paris; hpcd76, 78, 93, Henri Poincaré Archives, Nancy).

[22]Manuscript report of the Ph.D. thesis submitted by Henri Bouasse, 13 December, 1892, AJ[16]5535, Archives nationales, Paris. From Poincaré's conservative habits regarding formalism, he appears as an unlikely candidate at best for the development of a four-dimensional calculus circa 1905; cf. H. M. Schwartz's counterfactual conjecture: if Poincaré had adopted the ordinary vector calculus by the time he wrote his *Rendiconti* paper, "he would have in all likelihood introduced

explicitly … the convenient four-dimensional vector calculus" (1972, 1287, note 7).

[23] The invariants (1) may be expressed in ordinary vector notation, letting $\sum x = \mathbf{x}$, $\sum \xi = \mathbf{v}$, $\sum \xi_1 = \mathbf{v}_1$, and for convenience, $k = 1/\sqrt{1 - \sum \xi^2}$, $k_1 = 1/\sqrt{1 - \sum \xi_1^2}$, such that the four quantities (1) read as follows: $\mathbf{x}^2 - t^2$, $k(t - \mathbf{xv})$, $k_1(t - \mathbf{xv}_1)$, $kk_1(1 - \mathbf{vv}_1)$.

[24]Poincaré's four kinematic invariants (1) are functions of the following six intermediate invariants: $a = x^2 + y^2 + z^2 - t^2$, $b = x\delta x + y\delta y + z\delta z - t\delta t$, $c = x\delta_1 x + y\delta_1 y + z\delta_1 z - t\delta_1 t$, $d = \delta x\delta_1 x + \delta y\delta_1 y + \delta z\delta_1 z - \delta t\delta_1 t$, $e = \delta x^2 + \delta y^2 + \delta z^2 - \delta t^2$, $f = \delta_1 x^2 + \delta_1 y^2 + \delta_1 z^2 - \delta_1 t^2$. In terms of the latter six invariants, the four kinematic invariants (1) may be expressed as follows: $\sum x^2 - t^2 = a$, $A = -b/\sqrt{-e}$, $B = -c/\sqrt{-f}$, and $C = -d/(\sqrt{-e}\sqrt{-f})$. For a slightly different reconstruction of Poincaré's kinematic invariants, see Zahar (1989, 193).

[25]This definition was remarked by Pauli (1921, 637).

[26]The same subscript denotes the *force* acting on the *passive* mass, $\sum X_1$, and the *velocity* of the *active* mass, ξ_1.

[27]The ratio ρ/ρ' is equal to the Lorentz factor, since in Poincaré's configuration, $\varepsilon = -\xi$. Some writers hastily attribute a 4-current vector to Poincaré, the form $\rho(\xi,\ \eta,\ \zeta,\ i)$ being implied by Poincaré's 4-vector definitions of force density and velocity.

[28]"La transformation de Lorentz … agira sur ξ, η, ζ, 1 de la même manière que sur δx, δy, δz, δt, avec cette différence que ces expressions seront en outre multipliées par le *même* facteur $\delta t/\delta t' = 1/k(1 + \xi\varepsilon)$" (Poincaré 1906, 169).

[29]The invariants (4) may be expressed in ordinary vector notation, recalling the definitions of note 23, and letting $\sum X_1 = \mathbf{f}_1$, and $T_1 = \mathbf{f}_1\mathbf{v}$: $k^2\mathbf{f}_1^2(1 - \mathbf{v}^2)$, $k\mathbf{f}_1(\mathbf{x} - \mathbf{v}t)$, $kk_1\mathbf{f}_1(\mathbf{v}_1 - \mathbf{v})$, $k^2\mathbf{f}_1(\mathbf{v} - \mathbf{v})$. The fourth invariant is obviously null in this form.

[30]Poincaré's force invariants (4) are functions of the following six intermediate invariants: $m = k(X_1\delta x + Y_1\delta y + Z_1\delta z - T_1\delta t)$, $n = k(X_1\delta_1 x + Y_1\delta_1 y + Z_1\delta_1 z - T_1\delta_1 t)$, $o = k(X_1 x + Y_1 y + Z_1 z - T_1 t)$, $p = k^2(X_1^2 + Y_1^2 + Z_1^2 - T_1^2)$, $q = \delta x^2 + \delta y^2 + \delta z^2 - \delta t^2$, and $s = \delta_1 x^2 + \delta_1 y^2 + \delta_1 z^2 - \delta_1 t^2$. Let the four force invariants (4) be denoted by M, N, P, and S, then $M = p$, $N = o$, $P = n/\sqrt{-s}$, and $S = m/\sqrt{-q}$.

The same force invariants (4) are easily calculated using 4-vectors. Recalling the definitions in note 23 and note 29, let $\mathfrak{R} = (\mathbf{x}, \mathrm{i}t)$, $U = k(\mathbf{v}, \mathrm{i})$, $U_1 = k_1(\mathbf{v}_1, \mathrm{i})$, and $F_1 = k(\mathbf{f}_1, \mathrm{i}\mathbf{f}_1\mathbf{v})$, where $\sqrt{-1} = \mathrm{i}$. Then the force invariants (4) may be expressed as scalar products of 4-vectors: $M = F_1 F_1$, $N = F_1\mathfrak{R}$, $P = F_1 U_1$, and $S = F_1 U$.

[31]The kinematic invariants (1) obtained by Poincaré differ from those obtained from the products of 4-position and 4-velocity, contrary to Zahar's account (Zahar 1989, 194). Recalling the 4-vectors $\mathfrak{R}$, U, U_1 from note 30, we form the products: $\mathfrak{R}\mathfrak{R}$, $\mathfrak{R}U$, $\mathfrak{R}U_1$, and UU_1. In Poincaré's notation, the latter four products are expressed as follows:

$$\sum x^2 - t^2,\ -\frac{t - \sum x\xi}{\sqrt{1 - \sum \xi^2}},\ -\frac{t - \sum x\xi_1}{\sqrt{1 - \sum \xi_1^2}},\ -\frac{1 - \sum \xi\xi_1}{\sqrt{(1 - \sum \xi^2)\,(1 - \sum \xi_1^2)}}.$$

These invariants differ from those of Poincaré (1) only by the sign of A, B, and C, as noted by Sommerfeld (1910b, 686).

[32]An emission theory was proposed a few years later by Walter Ritz; see Ritz (1908).

[33]Using (5), Poincaré found the transformed force invariants $1/r_1^4$, $-1/r_1 - \sum x_1(\xi - \xi_1)/r_1^2$, and $\sum x_1(\xi - \xi_1)/r_1^3$.

[34]"Au premier abord, la solution (6) paraît la plus simple, elle ne peut néanmoins être adoptée; en effet, comme M, N, P sont des fonctions de X_1, Y_1, Z_1, et de $T_1 = \Sigma X_1\xi$, on peut tirer de ces trois équations (6) les valeurs de X_1, Y_1, Z_1; mais dans certains cas ces valeurs deviendraient imaginaires" (Poincaré 1906, 172).

[35]Replacing A and B in (6) by their definitions results in the three equations: $M = k^2\mathbf{f}_1^2(1 - \mathbf{v}^2) = 1/k^4(r + \mathbf{xv}_1)^4$, $N = \mathbf{f}_1(\mathbf{x} + \mathbf{v}r) = -(r + \mathbf{xv})/[k_1^2(r + \mathbf{xv}_1)^2]$, $P = kk_1\mathbf{f}_1(\mathbf{v}_1 - \mathbf{v}) =$

$[k(r + \mathbf{xv}) - k_1(r + \mathbf{xv}_1)]/[k_1^3(r + \mathbf{xv}_1)^3]$. Equations N and P imply an attractive force for all values of $\mathbf{v}$ and $\mathbf{v}_1$, while M leads to the ambiguously-signed solution: $\mathbf{f}_1 = \pm 1/[k^2(r + \mathbf{xv}_1)^2]$. Presumably, the superfluous plus sign in (6) is an indication of Poincaré's preoccupation with obtaining a force of correct sign.

[36] $A = -k_0(r + \sum x\xi)$, $B = -k_1(r + \sum x\xi_1)$, and $C = k_0 k_1(1 - \sum x\xi\xi_1)$.

[37] Using modern 4-vector notation, and denoting Poincaré's gravitational force 4-vector $F_1 = k_0(X_1, Y_1, Z_1, iT_1)$, equation (7) may be expressed: $F_1 = \alpha\mathfrak{R} + \beta U + \gamma U_1$, where $\mathfrak{R}$ denotes a lightlike 4-vector between the mass points, α, β, γ stand for undetermined functions of the three kinematic invariants A, B, and C, while $U = k_0(\mathbf{v}, i)$, $U_1 = k_1(\mathbf{v}_1, i)$ designate the 4-velocities of the passive and active mass points, respectively.

[38] In ordinary vector form, recalling the definitions in note 23 and note 29, the spatial part of Poincaré's law is expressed as follows: $\mathbf{f}_1 = -[(\mathbf{x}+r\mathbf{v}_1)+\mathbf{v}\times(\mathbf{v}_1\times\mathbf{x})]/[kk_1^3(r+\mathbf{xv}_1)^3(1-\mathbf{vv}_1)]$. Cf. Zahar (1989, 199).

[39] This law may be reformulated using the vectors defined in note 23 and note 29, and neglecting (with Poincaré) the component T_1: $\mathbf{f}_1 = -[(\mathbf{x} + r\mathbf{v}_1) + \mathbf{v} \times (\mathbf{v}_1 \times \mathbf{x})]/[k_1^2(r + \mathbf{xv}_1)^3]$. Cf. Zahar (1989, 199). Comparable expressions were developed by Lorentz (1910, 1239) and Kottler (1922, 169).

[40] "Alors λ, μ, ν, ou λ/B^3, μ/B^3, ν/B^3, est une espèce de champ électrique, tandis que λ', μ', ν', ou plutôt λ'/B^3, μ'/B^3, ν'/B^3, est une espèce de champ magnétique" (Poincaré 1906, 175).

[41] Whittaker (1951, 1:394, note 3). A 4-potential corresponding to Poincaré's second law (10) was given by Kottler (1922, 169). Additional assumptions are required in order to identify a "gravito-magnetic" field with a term arising from the Lorentz transformation of force: $\mathbf{v}\times(\mathbf{v}_1\times\mathbf{x})$, or the second term of the 3-vector version of (10) (neglecting the global factor; see note 39). In particular, it must be assumed that when the sources of the "gravito-electric" field $B^{-3}(\lambda, \mu, \nu)$ are at rest, the force on a mass point m is $\mathbf{f} = mB^{-3}(\lambda, \mu, \nu)$, independent of the velocity of m. For a detailed discussion, see Jackson (1975, 578).

[42] Poincaré reviewed Laplace's argument in his 1906–1907 lectures (Poincaré 1953, 194). For a contemporary overview of the question of the propagation velocity of gravitation see Tisserand (1889, 511).

[43] Fritz Wacker, a student of Richard Gans in Tübingen, published similar results in 1906.

[44] Poincaré (1908, 400). Poincaré explained to his students that Mercury's anomalous advance could plausibly be attributed to an intra-Mercurial matter belt (Poincaré 1953, 265), an idea advanced forcefully by Hugo von Seeliger in 1906 (Roseveare 1982, 78). In a lecture delivered in September, 1909, Poincaré revised his estimate of the relativistic perihelial advance downward slightly to 6" (Poincaré 1909).

[45] "[S]i nous admettions le postulat de relativité, nous trouverions dans la loi de gravitation et dans les lois électromagnétiques un nombre commun qui serait la vitesse de la lumière; et nous le retrouverions encore dans toutes les autres forces d'origine quelconque" (Poincaré 1906, 131).

[46] "Ou bien il n'y aurait rien au monde qui ne fût d'origine électromagnétique. Ou bien cette partie qui serait pour ainsi dire commune à tous les phénomènes physiques ne serait qu'une apparence, quelque chose qui tiendrait à nos méthodes de mesure" (Poincaré 1906, 131–132).

[47] Poincaré (1906, 132), Stein (1987, 397, note 29). On the history of magneto-cathode rays, see Carazza & Kragh (1990).

[48] "Il semble bien en effet qu'il serait possible de traduire notre physique dans le langage de la géométrie à quatre dimensions; tenter cette traduction ce serait se donner beaucoup de mal pour peu de profit, et je me bornerai à citer la mécanique de Hertz où l'on voit quelque chose d'analogue. Cependant, il semble que la traduction serait toujours moins simple que le texte, et qu'elle aurait toujours l'air d'une traduction, que la langue des trois dimensions semble la mieux appropriée à la description de notre monde, encore que cette description puisse se faire à la rigueur dans un autre

idiome" (Poincaré 1907, 15). See also Walter (1999b, 98), and for a different translation, Galison (1979, 95). On Hertz's mechanics, see Lützen (1999).

[49] "[D]ans les Sciences mathématiques, une bonne notation a la même importance philosophique qu'une bonne classification dans les Sciences naturelles" (Poincaré 1898, x).

[50] Poincaré's analysis of the concepts of space and time in relativity theory appeared in 1912 (Poincaré 1912). On the cool reception among mathematicians of Poincaré's views on physical geometry, see Walter (1997).

[51] Marcolongo (1906). This paper later gave rise to a priority claim for a slightly different substitution: $u = \mathrm{i}t$ (Marcolongo to Arnold Sommerfeld, 5 May, 1913, Archives for History of Quantum Physics 32). On Marcolongo's paper see also Maltese (2000, 135).

[52] Rüdenberg (1973), Serre (1993), Strobl (1985).

[53] Minkowski to Hilbert, 14 February, 1885, Rüdenberg & Zassenhaus (1973, 30). Minkowski's fears turned out to be for naught, as Poincaré pursued a different line of research (Zassenhaus 1975, 446). On Minkowski's early work on the geometry of numbers see Schwermer (1991); on later developments, see Krätzel (1989).

[54] See Gray (1992) and the reception study by Barrow-Green (1997, chap. 6).

[55] Minkowski to Hurwitz, 5 January, 1892, Cod. Ms. Math. Arch. 78: 188, Handschriftenabteilung, Niedersächsische Staats- und Universitätsbibliothek (NSUB). On Minkowski's report see also Barrow-Green (1997, 143).

[56] Minkowski papers, Arc. 4° 1712, Jewish National and University Library (JNUL); Minkowski to Hilbert, 11 March, 1901, Rüdenberg & Zassenhaus (1973, 139).

[57] Vorlesungen über analytische Mechanik, Wintersemester 1897/98, p. 29, Minkowski papers, Arc. 4° 1712, JNUL.

[58] Loc. cit. note 57. The concepts of scalar and vector mentioned by Minkowski were those introduced by W. R. Hamilton (1805–1865), the founder of quaternion theory. Even in Britain, vectors were judged superior to quaternions for use in physics, giving rise to spirited exchanges in the pages of *Nature* during the 1890s, as noted by Bork (1966) and Crowe (1967, chap. 6). On the introduction of vector analysis as a standard tool of the physicist during this period, see Jungnickel & McCormmach (1986, 2:342), and for a general history, see Crowe (1967).

[59] Klein (1897); Vorlesungen über Mechanik, Wintersemester 1898/99, 47, 59, Minkowski papers, Arc. 4° 1712, JNUL. Minkowski referred to Klein and Sommerfeld's text in relation to the concept of force and its anthropomorphic origins, the kinetic theory of gas, and the theory of elasticity.

[60] Minkowski to Sommerfeld, 30 October, 1898, MSS 1013A, Special Collections, National Museum of American History. An extensive reading list of mechanics texts is found in Minkowski's course notes for the 1903–1904 winter semester, Mechanik I, 9, Minkowski papers, Arc. 4° 1712, JNUL.

[61] Minkowski to Sommerfeld, 30 October, 1898, loc. cit. note 60; Minkowski to Sommerfeld, 18 November, 1899, Nachlass Sommerfeld, Arch HS1977-28/A, 233, Deutsches Museum München; research notebook, 12 December, 1899, Arc. 4° 1712, Minkowski papers, JNUL.

[62] Minkowski (1888, 1907); Minkowski to Sommerfeld, 30 October, 1898, loc. cit. note 60.

[63] On Göttingen's rise to preeminence in these fields, see Manegold (1970), Pyenson (1985b, chap. 7), and Rowe (1989, 1992).

[64] Nachlass Hilbert 570/9, Handschriftenabteilung, NSUB; Pyenson (1985b, chap. 5).

[65] *Jahresbericht der deutschen Mathematiker-Vereinigung* 14, 61.

[66] Although Poincaré spoke on celestial mechanics in Göttingen in 1895 (Rowe 1992, 475), and was invited back in 1902, he did not return until 1909, a few months after Minkowski's sudden death. See Hilbert to Poincaré, 6 November, 1908 (Dugac 1986, 209); Klein to Poincaré, 14 Jan., 1902 (Dugac 1989, 124–125). Sponsored by the Wolfskehl Fund, Poincaré's 1909 lecture

series took place during "Poincaré week", in the month of April. His lectures were published the following year (Poincaré 1910) in a collection launched in 1907, based on an idea of Minkowski's (Klein 1907, IV).

[67] *Jahresbericht der deutschen Mathematiker-Vereinigung* 14:128, 586; 15:154–155; 17:5.

[68] On Kaufmann's cathode-ray deflection experiments, see Miller (1981, 226) and Hon (1995). Readings of Kaufmann's articles are discussed at length by Richard Staley (1998, 270).

[69] Undated typescript of a lecture on a new form of the equations of electrodynamics, Math. Archiv 60:3, Handschriftenabteilung, NSUB. This typescript differs significantly from the posthumously-published version (1915).

[70] "Es entsteht die grosse Frage, wie sich denn das Gravitationsgesetz in das Reich des Relativitätsprinzipes einordnen lässt" (p. 15).

[71] Actually, Poincaré postulated the lightlike propagation velocity of gravitation, as mentioned above(p. 10).

[72] "Poincaré weist ein solches Gesetz auf, indem er auf die Betrachtung von Invarianten der Lorentzschen Gruppe eingeht, doch ist das Gesetz nur eines unter vielen möglichen, und die betreffenden Untersuchungen tragen in keiner Weise einen definitiven Charakter" (p. 16). See also Pyenson (1973, 233).

[73] *Jahresbericht der deutschen Mathematiker-Vereinigung* 17 (1908), Mitt. u. Nachr., 4–5.

[74] The same term was employed by Cayley to denote a line which meets any given lines, in a paper of 1869.

[75] When written out in full, one obtains, for example, $\psi_{23} = \partial\psi_3/\partial x_2 - \partial\psi_2/\partial x_2 = \mathfrak{h}_x$. Minkowski later renamed the Traktor a *Raum-Zeit-Vektor II. Art* (Minkowski 1908, § 5), but it is better known as either a 6-vector, an antisymmetric 6-tensor, or an antisymmetric, second-rank tensor. As the suite of synonyms suggests, this object found frequent service in covariant formulations of electrodynamics.

[76] For a comparison of the Lorentz and Larmor theories, see Darrigol (1994).

[77] On the four-dimensional transcription of Ohm's law see Arzeliès & Henry (1959, 65–67).

[78] Minkowski mentions this very property on p. 6.

[79] The passage in question may be translated as follows: "Next we consider $X, Y, Z, T\sqrt{-1}$, as the coordinates of a fourth point Q; the invariants will then be functions of the mutual distances of the five points O, P, P', P'', Q, and among these functions we must retain only those that are 0th degree homogeneous with respect, on one hand, to X, Y, Z, T, δx, δy, δz, δt (variables that can be further replaced by X_1, Y_1, Z_1, T_1, ξ, η, ζ, 1), and on the other hand, with respect to $\delta_1 x$, $\delta_1 y$, $\delta_1 z$, 1 (variables that can be further replaced by ξ_1, η_1, ζ_1, 1)" (Poincaré 1906, 170). The misprint is in the next-to-last set of variables, where instead of 1 we should have $\delta_1 t$.

[80] One other obvious source for Minkowski's error is Lorentz's transformation of charge density: $\varrho' = \varrho/\beta l^3$, where $1/\beta = \sqrt{1 - v^2/c^2}$, and l is a constant later set to unity (Lorentz 1904a, 813), although this formula was carefully corrected by Poincaré.

[81] Let the differential parameter $d\tau$ of a worldline be expressed in Minkowskian coordinates by $d\tau^2 = -(dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2)$. The 4-velocity vector U_μ is naturally defined to be the first derivative with respect to this parameter, $U_\mu = dx_\mu/d\tau$ ($\mu = 1, 2, 3, 4$).

[82] The incongruity noted by Pyenson (1985b, 84) between Minkowski's announcement of a four-dimensional physics on one hand, and on the other hand, a trifle of 4-vector definitions and expressions, is to be understood as a indication of Minkowski's gradual ascent of the learning curve of four-dimensional physics.

[83] Minkowski defined the spatial components of the empty space force density 4-vector $\mathfrak{X}_j$ in terms of the ordinary force density components $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$, and their product with velocity: $\mathfrak{A} = \mathfrak{X}\mathfrak{w}_x, \mathfrak{Y}\mathfrak{w}_y, \mathfrak{Z}\mathfrak{w}_z$, such that $\mathfrak{X}_j = \mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}, \mathrm{i}\mathfrak{A}$. He also expressed the force density 4-vector as the product of 4-current-density and the Traktor: $\mathfrak{X}_j = \varrho_1\psi_{j1} + \varrho_2\psi_{j2} + \varrho_3\psi_{j3} + \varrho_4\psi_{j4}$.

[84]Minkowski's manuscript was delivered to the printer on 21 February, 1908, corrected, and published on 5 April, 1908 (Journal für die "Nachrichten" der Gesellschaft der Wissenschaften zu Göttingen, mathematische-naturwissenschaftliche Klasse 1894–1912, Scient. 66, Nr. 1, 471, Archiv der Akademie der Wissenschaften zu Göttingen). I thank Tilman Sauer for pointing out this source to me.

[85]While the published version of Minkowski's 5 November lecture refers on one occasion to a "Raumzeitpunkt" (Minkowski 1915, 934), the term occurs nowhere in the archival typescript. The source of this addition is unknown. A manuscript annotation of the first page of the typescript bears Sommerfeld's initials, and indicates that he compared parts of the typescript to the proofs, as Lewis Pyenson (1985b, 82) points out. Pyenson errs, however, in attributing to Sommerfeld the authorship of the remaining annotations, which were all penned in Minkowski's characteristic cramped hand.

[86]While Minkowski's tensor is traceless, it is also asymmetric, a fact which led to criticism and rejection by leading theorists of the day. His asymmetric tensor was later rehabilitated; for a technical discussion with reference to the original papers, see Møller (1972, 219). In the absence of matter, his tensor assumes a symmetric form; in this form, it was hailed by theorists.

[87]Minkowski defined the energy-momentum tensor S in two ways: as the product of six-vectors, $fF = S - L$, where L is the Lagrange density, and in component form via the equations for Maxwell stresses, the Poynting vector, and electromagnetic energy density (Minkowski 1908, 96).

[88]Minkowski's alternative between a 4-force definition and the "natural" spacetime equations of motion was underlined by Pauli (1921, 664).

[89]Planck (1906, eqn. 6), Minkowski (1909, §4). In the latter lecture, Minkowski proposed the modern definition of kinetic energy as the temporal component of 4-momentum times c^2, or $mc^2 dt/d\tau$. The "spatial" part of the driving force (21) was referred to by Lorentz (1910, 1237) as a "Minkowskian force" (*Minkowskische Kraft*), differing from the Newtonian force by a Lorentz factor. Lorentz complemented the Minkowskian force with a "Minkowskian mass" (*Minkowskische Masse*).

[90]Minkowski's argument may be summarized as follows. From the definition of a 4-vector, the following orthogonality relation holds for the driving force R:

$$R_x \frac{dx}{d\tau} + R_y \frac{dy}{d\tau} + R_z \frac{dz}{d\tau} = R_t \frac{dt}{d\tau}. \tag{44}$$

Integration of the rest-mass density over the hypersurface normal to the worldline of the mass point results in the driving force components (21), but if the integration is to be performed instead over a constant-time hypersurface, proper time is replaced by coordinate time, such that the fourth equation reads: $md/dt(dt/d\tau) = R_t d\tau/dt$. From (44) we obtain an expression for R_t, which we multiply by $d\tau/dt$:

$$m \frac{d}{dt}\left(\frac{dt}{d\tau}\right) = \mathfrak{w}_x R_x \frac{d\tau}{dt} + \mathfrak{w}_y R_y \frac{d\tau}{dt} + \mathfrak{w}_z R_z \frac{d\tau}{dt}. \tag{45}$$

Minkowski reasoned that since the right-hand side of (45) describes the rate at which work is done on the particle, the left-hand side must be the rate of change of the particle's kinetic energy, such that (45) represents the law of energy conservation. He immediately related (45) to the kinetic energy of the particle:

$$m\left(\frac{dt}{d\tau} - 1\right) = m\left(\frac{1}{\sqrt{1-\mathfrak{w}^2}} - 1\right) = m(\tfrac{1}{2}|\mathfrak{w}|^2 + \tfrac{3}{8}|\mathfrak{w}|^4 + \cdots). \tag{46}$$

Minkowski did not justify the latter expression, but in virtue of his definition of proper time, $d\tau = dt\sqrt{1-\mathfrak{w}^2}$, the left-hand side of (45) may be rewritten as $m(d/dt)(1/\sqrt{1-\mathfrak{w}^2})$, such that upon integration the particle's kinetic energy is $m/\sqrt{1-\mathfrak{w}^2}+C$, where C is a constant. For agreement with the Newtonian expression of kinetic energy in case of small particle velocities ($\mathfrak{w} \ll 1$), we let $C=-m$, which accords both with (46) and the definition of kinetic energy given in a later lecture (cf. note 89).

[91] *Jahrbuch über die Fortschritte für Mathematik* 39, 1908, 910.

[92] This trend is described by Darrigol (1993, 270). The sharp contrast between the importance assigned to vector methods in France and Germany may be linked to the status accorded to applied mathematics in these two nations, as discussed by H. Gispert in her review of the French version of Klein's *Encyklopädie* (Gispert 2001).

[93] At one point during his calculations Minkowski seemed convinced of the utility of this formalism, remarking that electrodynamics is "predestined for application of quaternionic calculations" (Math. Archiv 60:6, 21, Handschriftenabteilung, NSUB).

[94] A precedent for Minkowski's exclusive use of lor may be found in Gibbs & Wilson (1901), where ∇ is similarly preferred to vector functions.

[95] Cf. Max von Laue's remark that physicists understood little of Minkowski's work because of its unfamiliar mathematical expression (Von Laue 1951, 515), and Chuang Liu's account of the difficulty experienced by Max Abraham and Gunnar Nordström in applying Minkowski's formalism (Liu 1991, 66). While Minkowski's calculus is a straightforward extension of Cayley's formalism (for a summary, see Cunningham 1914, chap. 8), the latter formalism was itself unfamiliar to physicists.

[96] Math. Archiv 60:5, Handschriftenabteilung, NSUB. This 82-page set of notes dates from 23 May to 6 October, 1908. A posthumously published paper on the electron-theoretical derivation of the basic equations of electrodynamics for moving media, while purported to be from Minkowski's Nachlass, was written entirely by Max Born, as he acknowledged (Minkowski & Born 1910, 527). In the latter publication lor makes only a brief appearance.

[97] Math. Archiv 60:6, 10, Handschriftenabteilung, NSUB.

[98] This "peculiar" consequence of Minkowski's spacetime mechanics was underlined by Maxwell's German translator, the Berlin physicist Max B. Weinstein (1914, 42). In Minkowski spacetime, 4-acceleration is orthogonal to 4-velocity: $U_\mu dU_\mu/d\tau=0$, $\mu=1, 2, 3, 4$, where τ is the proper time. We assume a 4-scalar potential Φ such that the gravitational 4-force $F_\mu=-m\partial\Phi/\partial x_\mu$. If we consider a point mass with 4-velocity U_μ subjected to a 4-force F_μ derived from this potential, we have $U_\mu F_\mu=-U_\mu m\partial\Phi/\partial x_\mu$. Writing 4-velocity as $dx_\mu/d\tau$, and substituting in the latter expression, we obtain

$$U_\mu F_\mu=-m\frac{dx_\mu}{d\tau}\frac{\partial\Phi}{\partial x_\mu}=-m\frac{d\Phi}{d\tau}=0,$$

and consequently, $d\Phi/d\tau=0$, which means that the law of motion describes the trajectory of the passive mass m only in the trivial case of constant Φ along its worldline.

[99] Minkowski introduced the terms *zeitartig* and *raumartig* in (1909).

[100] There is little agreement on where to situate Minkowski's work on relativity along a line running from the intuitive to the formal. Peter Galison (1979, 89), for example, underlines Minkowski's visual thinking (i.e., reasoning that appeals to figures or diagrams), while Leo Corry (1997, 275; 2004, chap. 4) considers Minkowski's work in the context of Hilbert's axiomatic program for physics.

[101] Two spatial dimensions are suppressed in Figure 1, and lightcones are represented by broken lines with slope equal to ±1, the units being chosen so that the propagation velocity of light is unity ($c=1$). In this model of Minkowski space, orthogonal coordinate axes appear oblique in general, for example, the spatial axes $x{*}y{*}z{*}$ are orthogonal to the tangent $B{*}C{*}$ at spacetime point $B{*}$ of the central line of the filament $F{*}$ described by a particle of proper mass $m{*}$.

[102]The 4-vector OA' in (24) has unit magnitude by definition in all inertial frames, while $B{*}D{*}$ is a timelike 4-vector tangent to the central line of $F{*}$ at $B{*}$. Consequently, $B{*}D{*}$ may be taken to coincide with the temporal axis of a frame instantaneously at rest with $m{*}$ at $B{*}$, such that it has only one nonzero component: the difference in proper time between the points $B{*}$ and $D{*}$. It is assumed that the rest frame may be determined unambiguously for a particle in arbitrary motion, as asserted without proof by Minkowski in a later lecture (Minkowski 1909, § III); subsequently, Max Born (1909, 26) remarked that any motion may be approximated by what he called hyperbolic motion, and noted that such motion is characterized by an acceleration of constant magnitude (as measured in an inertial frame). If we locate the origin of this frame at $B{*}$, and let $D{*} = (0, 0, 0, t)$, then $B{*}D{*} = (0, 0, 0, \mathrm{i}t)$, and $(B{*}D{*})^3 = -\mathrm{i}t^3$. Likewise in this same frame, $A = A' = (0, 0, 0, 1)$, and $OA' = OA = (0, 0, 0, \mathrm{i})$. Minkowski understood the term $(OA'/B{*}D{*})$ as the ratio (*Verhältnis*) of two parallel 4-vectors, an operation familiar from the calculus of quaternions, but one not defined for 4-vectors. While modern vector systems ignore vector division, in Hamilton's quaternionic calculus the quotient of vectors is unambiguously defined; see, for example, Tait (1882, chap. 2). Accordingly, the quotient in (24) is the ratio of lengths, $(OA'/B{*}D{*}) = 1/t$, and the cubed ratio is t^{-3}. The point B lies on the same constant-time hypersurface as $D{*}$, so we assign it the value $(x, y, z, t) = (\mathbf{r}, t)$. This assignment determines the value of the 4-vector $BD{*}$: $B{*}D{*} = (-x, -y, -z, 0) = (-\mathbf{r}, 0)$. Since $B{*}$ is a lightpoint of B, we can apply (23) to obtain $x^2 + y^2 + z^2 = t^2 = r^2$, and consequently, $t^3 = r^3$. Substituting for t^3 results in $(OA'/B{*}D{*})^3 = 1/t^3 = 1/r^3$. The 4-vector $B{*}D{*}$ is spacelike, such that its projection on the constant-time hypersurface orthogonal to $B{*}D{*}$ at $D{*}$ is the ordinary vector $(-x, -y, -z) = -\mathbf{r}$. In terms of ordinary vectors and scalars measured in the rest frame of $m{*}$, Minkowski's expression (24) is equivalent to Newton's law (neglecting the gravitational constant):

$$- mm{*}\frac{\mathbf{r}}{r^3}. \tag{47}$$

Neither (24) nor (47) contains any velocity-dependent terms, while the timelike component of Minkowski's first law depends on the velocity of the passive mass. Newton's law (47) thus coincides with Minkowski's first law only in the case of relative rest.

[103]Minkowski's driving force may be expressed in his notation as a function of scalar products of 4-velocities and 4-position:

$$-mm{*}\frac{(w\bar{w}{*})\mathfrak{R} - (w\bar{\mathfrak{R}})w{*}}{(\mathfrak{R}\bar{w}{*})^3(w\bar{w}{*})}.$$

Here I let w and $w{*}$ designate 4-velocity at the passive and active mass points, while $\mathfrak{R}$ is the associated 4-position, the parentheses denote a scalar product, and the bar indicates transposition.

[104]The intermediate calculations can be reconstructed as follows. Let the driving force be designated F_μ, $\mu = 1, 2, 3, 4$. Since $(OA'/B{*}D{*})^3 = t^{-3}$, and $BD{*} = (-x, -y, -z, 0)$, equations (21) and (24) yield: $F_1/m = d^2x/d\tau^2 = -m{*}x/(t - \tau{*})^3$, $F_2/m = d^2y/d\tau^2 = -m{*}y/(t - \tau{*})^3$, $F_3/m = d^2z/d\tau^2 = -m{*}z/(t - \tau{*})^3$.

[105]Minkowski omitted the intermediate calculations, which may be reconstructed in modern notation as follows. Let the 4-velocity of the passive mass point be designated $U_\mu = (dx/d\tau, dy/d\tau, dz/d\tau, \mathrm{i}dt/d\tau)$, while the first three components of its 4-acceleration, designated A_μ, at B due to the source $m{*}$ are given by (26). From the orthogonality of 4-velocity and 4-acceleration we have:

$$U_\mu A_\mu = -\frac{dx}{d\tau}\frac{m{*}x}{(t - \tau{*})^3} - \frac{dy}{d\tau}\frac{m{*}y}{(t - \tau{*})^3} - \frac{dz}{d\tau}\frac{m{*}z}{(t - \tau{*})^3} - \frac{\mathrm{i}dt}{d\tau}\frac{\mathrm{i}d^2t}{d\tau^2} = 0. \tag{48}$$

Rearranging (48) results in an expression for the temporal component of 4-acceleration:

$$\frac{d^2t}{d\tau^2} = -\frac{m{*}}{(t - \tau{*})^3}\left(\frac{xdx}{dt} + \frac{ydy}{dt} + \frac{zdz}{dt}\right). \tag{49}$$

Differentiating (25) with respect to dt results in $xdx/dt+ydy/dt+zdz/dt = (t-\tau^*)d(t-\tau^*)/dt$, the right-hand side of which we substitute in (49) to obtain (27).

[106]A young Polish physicist in Göttingen, Felix Joachim de Wisniewski later studied this case, but with equations differing from (26) by a Lorentz factor (Wisniewski 1913a, 388). In a postscript to the second installment of his paper (Wisniewski 1913b, 676), he employed Minkowski's matrix notation, becoming, with Max Born, one of the rare physicists to adopt this notation.

[107]The intermediate calculations were omitted by Minkowski, but figure among his research notes (Math. Archiv 60:6, 126–127, Handschriftenabteilung, NSUB). Following the method outlined by Otto Dziobek (1888, 12), Minkowski began with the energy integral of Keplerian motion:

$$\left(\frac{dt}{dW}\right)^2 - 1 = \frac{2}{\ell^2}\left(\frac{M}{R} - C\right), \tag{50}$$

where ℓ denotes the velocity of light, M is the sum of the masses times the gravitational constant, $M = k^2(m + m^*)$, R is the radius, and C is a constant. The left-hand side of (50) is the same as the right-hand side of (29) for $W = \tau$. In order to express dt/dW (which is to say $dt/d\tau$) in terms of E, Minkowski considered a conic section in polar coordinates, with focus at the origin: $R = a(1-e^2)/(1+e\cos\varphi) = a(1-e\cos E)$, where a denotes the semi-major axis, and φ is the true anomaly. By eliminating φ in favor of E and e, and differentiating (28), Minkowski obtained an expression equivalent to (30).

[108]I insert the eccentricity e in the second term on the right-hand side, correcting an obvious omission in Minkowski's paper (1908, 111, eqn. 31).

[109] "…eine Entscheidung *gegen* ein solches Gesetz und die vorgeschlagene modifizierte Mechanik zu Gunsten des Newtonschen Attraktionsgesetzes mit der Newtonschen Mechanik aus den astronomischen Beobachtungen nicht abzuleiten sein" (Minkowski 1908, 111).

[110]Minkowski's explanation of the construction of his spacetime diagram (Figure 3) may be paraphrased in modern terminology as follows. Suppressing the z-axis, we associate two worldlines with two point charges e_1 and e. The worldline of e_1 passes through the point at which we wish to determine the field, P_1. To find the retarded position of the source e, we draw the retrograde lightcone (with broken lines) from P_1, which intersects the worldline of e at P, where there is a hyperbola of curvature ϱ with three infinitely-near points lying on the worldline of e; it has its center at M. The coordinate origin is established at P, by letting the t-axis coincide with the tangent to the worldline. A line from P_1 intersects this axis orthogonally at point Q; it is spacelike, and if its projection on a constant-time hypersurface has length r, the length of the 4-vector PQ is r/c. The 4-vector potential has magnitude e/r and points in the direction of PQ (i.e., parallel to the 4-velocity of e at P). The x-axis lies parallel to QP_1, such that N is the intersection of a line through M normal to the x-axis.

[111]Minkowski noted four conditions on $\mathfrak{K}$: it is normal to the 4-velocity of e_1 at P_1, $c\mathfrak{K}_t - \mathfrak{K}_x = 1/r^2$, $\mathfrak{K}_y = \ddot{y}/(c^2 r)$, and $\mathfrak{K}_z = 0$, where r is the spacelike distance between the test charge e_1 at P_1 and the advanced position Q of the source e, and $\ddot{y}$ is the y-component of e's 4-acceleration at P. For a derivation of the 4-potential and 4-force corresponding to Minkowski's presentation, see Pauli (1921, 644–645).

[112]See § 1, p. 5,as well as Heaviside (1893), and Gans (1905). Theories in which the gravitational field is determined by equations having the form of Maxwell's equations were later termed vector theories of gravitation by Max Abraham (1914, 477). For a more recent version of such a theory, see Coster & Shepanski (1969).

[113]Minkowski's neglect of the differences between his two theories may explain why historians have failed to distinguish them. The principal difference between the two laws stems from the presence of acceleration effects in the second law. By 1905 it was known that accelerated electrons radiate energy, such that by formal analogy, a Maxwellian theory of gravitation should

have featured accelerated point masses radiating "gravitational" energy. For a brief overview of research performed in the first two decades of the twentieth century on the energy radiated from accelerated electrons, see Whittaker (1951, 2:246).

[114]"In der dem Weltpostulate gemäß reformierten Mechanik fallen die Disharmonien, die zwischen der Newtonschen Mechanik und der modernen Elektrodynamik gestört haben, von selbst aus" (Minkowski 1909, § 5).

[115]Abraham showed that a mass set into oscillation would be unstable due to the direction of energy flow (Norton 1992, 33). On the early history of unified field theories, see the reference in note 12.

[116]Sommerfeld's work on the *Encyklopädie* is discussed in an editorial note to his scientific correspondence (Eckert & Märker 2001, 40).

[117]Reiff & Sommerfeld (1904), Lorentz (1904c, 1904b), Sommerfeld (1904a). The scheme proposed by Sommerfeld differed from that published in articles 12 to 14 of the *Encyklopädie* only in that the operands of scalar and vector products were no longer separated by a dot.

[118]Sommerfeld to Lorentz, 21 March, 1901, Eckert & Märker (2001, 191). On Sommerfeld's participation on the Commission see Reich (1996) and Eckert & Märker (2001, 144).

[119]For bibliometric data, and discussions of Sommerfeld's role in the rise of relativity theory, see Walter (1999a, 68–73; 1999b, 96, 108).

[120]See Abraham (1910), Lewis (1910a, 1910b), Varičak (1910). On Varičak's contribution see Walter (1999b).

[121]Not all of Sommerfeld's notational choices were retained by later investigators; Laue, for instance, preferred a notational distinction between 4-vectors and 6-vectors. For a summary of notation used by Minkowski, Abraham, Lewis, and Laue, see Reich (1994).

[122]"Mit Umkehr des bei Poincaré offenbar versehentlichen Vorzeichens" (Sommerfeld 1910b, 686, note 1).

[123]Sommerfeld explained Figure 4 roughly as follows: two skew 4-velocities $\mathfrak{B}$ and $\mathfrak{B}_0$ determine a three-dimensional space, containing all the lines shown. Points $OLSAP$ are coplanar, while the triangles OQT and OTS, and the parallelogram $LQTS$ all generally lie in distinct planes. In particular, T lies outside the plane of $OLSAP$, and OT is orthogonal to $\mathfrak{B}_0$. The broken vertical line l represents the temporal axis of a frame with origin O; a spacelike plane orthogonal to l at O intersects the worldline of m at point A. The spacelike 4-vector $\mathfrak{R}'$ is orthogonal to $\mathfrak{B}$, while $\mathfrak{S}$ is orthogonal to $\mathfrak{B}_0$; both $\mathfrak{R}'$ and $\mathfrak{S}$ intersect the origin, while $\mathfrak{B}$ and $\mathfrak{B}_0$ together form an angle ψ.

[124]This view was confirmed independently by the Dutch astronomer W. de Sitter, who worked out the numbers for the one-body problem (De Sitter 1911). De Sitter found the second law to require a post-Newtonian centennial advance in Mercury's perihelion of 7", while the first law required no additional advance. His figure for the second law agrees with the one given by Poincaré(above, p. 14).

[125] "In dieser und einigen anschließenden Studien möchte ich darstellen, wie merkwürdig sich die elektrodynamischen Begriffe und Rechnungen vereinfachen, wenn man sich dabei von der tiefsinnigen Raum-Zeit-Auffassung Minkowskis leiten läßt" (Sommerfeld 1910a, 749).

[126]Faced with a similar situation in his Cologne lecture of September, 1908, Minkowski simply neglected to mention Poincaré's contribution; see Walter (1999a, 56).

[127]Sommerfeld later preferred Gustav Mie's field theory of gravitation. Such an approach was more promising than that of Poincaré and Minkowski, which grasped gravitation "to some extent as action at a distance" (Sommerfeld 1913, 73).

[128]"Die geometrische Systematik ist jetzt hyper-minkowskisch" (Sommerfeld to Wien, 11 July, 1910; Eckert & Märker 2001, 388).

[129]Einstein to Sommerfeld, July, 1910, Klein (1993, 243–247); Eckert & Märker (2001, 386–388). In light of Einstein and Laub's earlier dismissal of Minkowski's formalism (see above, p. 26), Sommerfeld naturally supposed that Einstein would disapprove of his system, prompting the protest: "Wie können Sie denken, dass ich die Schönheit einer solchen Untersuchung nicht zu schätzen wüsste?"

[130]"[Minkowski] hat auf seine Darstellung vielleicht ebenso viel Wert gelegt, wie auf ihren Inhalt. Darum glaube ich nicht, daß der Zugang zu seiner Gedankenwelt erleichtert wird, wenn sie von einer ungeheuren [sic] Fülle von Formeln überschüttet wird" (Born 1914).

[131]By the end of 1911 Born had already acknowledged that, despite its "formal simplicity and greater generality compared to the tradition of vectorial notation," Minkowski's calculus was "unable to hold its ground in mathematical physics" (Born 1912, 175).

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