

The Non-Euclidean Style of Minkowskian Relativity

Scott A. Walter*

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1 Introduction

The history of relativity is structured for most commentators by two landmark discoveries due to Albert Einstein: the special theory (1905) and the general theory of relativity (1915). To get from one theory to the other, we know that Einstein relied on a certain number of fundamental concepts, such as the equivalence principle, and a few key mathematical techniques, for instance, the absolute tensor calculus of Gregorio Ricci-Curbastro and Tullio Levi-Civita. Einstein also had need of a third theory and technique, elaborated by his former mathematics professor, Hermann Minkowski (1864–1909), although he did not recognize this for several years. In this paper, we examine the fortunes of Minkowski’s space-time theory from 1908 to 1916. Our focus is on the emergence of Minkowski’s four-dimensional formalism as a standard technique in theoretical physics, and we investigate one aspect of this history in some detail: the reformulation and reinterpretation of the laws of special relativity in the language of non-Euclidean geometry. The related work done on the space-time theory, or what we call the “non-Euclidean style” of Minkowskian relativity, provides an example of the geometrization of physics brought about by Minkowski and his followers.

In order to situate our topic in a broader scientific context, we first describe the status of applications of non-Euclidean geometry in physics around the turn of the century. Next, we present a quantitative overview of publications on Minkowskian relativity for the period 1908–1915. We then review Minkowski’s appeal to non-Euclidean geometry, and link this to the mixed reception of his work. There follows a comparative study of the emergence and development of the non-Euclidean style in selected works by Arnold Sommerfeld, Alfred A. Robb, Vladimir Varičak, Gilbert N. Lewis, Edwin B. Wilson and Émile Borel.

2 Pre-Minkowskian applications of non-Euclidean geometry in physics

At the end of the nineteenth century, several mathematicians showed an interest in applying non-Euclidean geometry to physics. The titles listed in Duncan Sommerville’s 1911 bibliography of non-Euclidean and n -dimensional geometry give one an idea of the level of activity in

*University of Nantes, Centre François Viète

this area. For the period from 1890 to 1905, we find a total of forty-nine titles on kinematics or dynamics in non-Euclidean space,¹ to be compared with a total of over two thousand titles covering all aspects of non-Euclidean and n -dimensional geometry published during the same period.

The title count in Sommerville's bibliography points to a modest trend of physical applications of non-Euclidean geometry, but says little of mathematicians' attitudes toward the physical significance of non-Euclidean and n -dimensional geometry. According to a well-known doctrine formulated by the French mathematician Henri Poincaré (1854–1912), the geometry realized in physical space can not be determined in an unambiguous fashion. The axioms of geometry are not synthetic *a priori* judgments, as Kant believed, but freely-stipulated conventions. However, all conventions are not equal. "Euclidean geometry," Poincaré insisted, "is and will remain the most convenient."²

Over the years, several commentators (including Jammer 1960, 163, and Kline 1972, 922) have considered Poincaré's doctrine as the dominant one among turn-of-the-century mathematicians. Yet not a single geometer supported Poincaré's extreme position on the nature of space. Anti-conventionalists included Jacques Hadamard and Émile Picard in France, Federigo Enriques, Gino Fano and Francesco Severi in Italy, Heinrich Liebmann, Eduard Study, Aurel Voss and David Hilbert in Germany. According to these mathematicians, the geometry of space was subject in principle to empirical determination, just as Helmholtz and other physicists had claimed (Walter 1997, 104).

As for the claim that Euclidean geometry would forever remain the most convenient, the theoretical physicists Ernst Mach and Ludwig Boltzmann both implicitly took exception to Poincaré's assumption that the laws of physics could (and would) be adjusted in order to save Euclidean geometry (Walter 1997, 110-111). The convenience of non-Euclidean geometry for investigations in certain domains of pure mathematics, on the other hand, was widely acknowledged by mathematicians by 1900. As Christian Houzel observes (1991, 179), Poincaré's use of hyperbolic geometry to demonstrate the existence of Fuchsian functions in 1880 was path-breaking in this regard (see Gray & Walter 1997).

Some of the abstract questions which lent themselves to the techniques of non-Euclidean geometry had strong links to the problems of physics. Perhaps the best-known example of a crossover of this sort is Hertz's mechanics. While he assumed material points to move in Euclidean space and absolute time, Hertz applied variational methods in an n -dimensional configuration space, in which the number of dimensions corresponds to the degrees of freedom of the system under investigation. This geometrical interpretation of Hamiltonian mechanics, however, was criticized for its hypothetical nature and scant results by both Poincaré and Boltzmann.³ Interest in such efforts abided nonetheless; in particular, readers appreciated the sophistication of the methods employed to solve dynamics problems in non-Euclidean spaces of n dimensions.⁴

Starting in the 1890s, the multiplication of university courses dedicated to non-Euclidean geometry fostered the diffusion of mathematical techniques used in this area. The universities of Göttingen, Cambridge, and Johns Hopkins offered lecture courses of this sort at the end of the nineteenth century. In the years 1902-1904, according to listings in the *Jahresbericht*

¹This subject matter corresponds to category R in the then-standard classification scheme.

²"La géométrie euclidienne est et restera la plus commode" (Poincaré 1902, 76).

³Poincaré 1897, 743; Poincaré 1907, 15; Boltzmann 1905, 329. As Jesper Lützen points out (Lützen 1995b, 69-70), Hertz himself denied any practical value to his mechanics.

⁴On the interactions between mechanics and differential geometry in this period, see Ziegler (1985) and Lützen (1995a).

der deutschen Mathematiker-Vereinigung, five German universities offered courses on non-Euclidean geometry: Leipzig, Greifswald, Münster, Marburg and Königsberg.

In connection with these courses, geometers published textbooks outlining the history and formal development of mathematical methods of non-Euclidean geometry, which likewise favored the dissemination of knowledge in this domain. Here non-Euclidean geometry was presented as a unified intellectual field, which could be approached from three principal directions: projective geometry, differential geometry, and axiomatics. The emphasis given to any one approach varied substantially from place to place. The Göttingen mathematician Felix Klein (1849–1925), for example, in his 1889–1890 lectures on non-Euclidean geometry, elaborated projective methods in great detail, wasting little time on other approaches (Klein 1890–1892). Even so, Klein saw a fundamental unity in the subject of non-Euclidean geometry. Rather than a heterogeneous collection of abstruse mathematics, non-Euclidean geometry was in Klein’s view a “concrete discipline” (*reale Disziplin*).⁵

More balanced than Klein’s text are the later books by Heinrich Liebmann (1905) and Roberto Bonola (1906), which include chapters on hyperbolic trigonometry, Cayley geometry, differential geometry and axiomatics. For Liebmann, Bonola and others, the techniques of projective geometry, differential geometry and the axiomatic method were unified by their object, and in this sense, their writings contributed toward the intellectual unification of this emergent subdiscipline.

This unified image was also propagated by lecturers at scientific meetings. An oft-cited example is the glowing report on the state of hyperspace and non-Euclidean mechanics (including that of Hertz), read by the Kiel mathematician Paul Stäckel (1862–1919) at the the 1903 meeting of the German Association in Kassel. Far from the idle mathematical investigation of abstruse details some believed it to be, Stäckel considered the development of applications of a general mechanics to different branches of physics to hold great promise.⁶

In summary, by the first years of the twentieth century, non-Euclidean geometry had found a respectable place in the mathematics curriculum of several German universities, while the techniques of non-Euclidean geometry were further diffused on an elementary level through textbooks, which often portrayed the mechanics of non-Euclidean space as the very horizon of mathematical research. In contrast to the amount of publicity they received, applications of non-Euclidean geometry to physics by leading practitioners produced slim theoretical results, the value of which was outstripped by the technical intricacy of the methods deployed to obtain them. Nevertheless, some mathematicians and theoretical physicists continued to study and develop these methods and applications for their intrinsic interest.

3 A quantitative view of two geometrical approaches to relativity theory

When the first papers on the principle of relativity appeared in 1905, physicists generally presented their results in mechanics and electrodynamics using either Cartesian coordinate or vector methods. It was also about 1905 when Oliver Heaviside’s vector calculus became popular

⁵Klein (1890, 571), quoted in Hawkins (1980, 319). Jeremy Gray (1989, 155) gives an accessible account of Klein’s projective model of elliptic geometry. David Rowe (1989) considers Klein’s promotion of a group-theoretical approach to geometry and mechanics as one of several actions designed to adapt mathematics to the demands of the German industrial economy.

⁶Stäckel 1903, 481. Similar sentiment is expressed in a variety of texts on non-Euclidean geometry, for example, Barbarin 1902, §8 ; Liebmann 1905, §55; Wellstein 1905, §14; Bonola 1906, App. I.

among germanophone electrodynamicists (Reich 1996, 205), who naturally employed the same formalism in their studies of the principle of relativity.

In the spring of 1908, Hermann Minkowski published a new four-dimensional matrix formalism designed to take full advantage of the known covariance of physical laws with respect to the Lorentz group. Physicists developed a four-dimensional vector and tensor formalism on the basis of Minkowski's work, which we refer to as the space-time formalism. By this term we mean a four-dimensional calculus in which the temporal coordinate is imaginary and treated on an equal footing with the real spatial coordinates.

Applications of hyperbolic functions to relativity constitute the non-Euclidean style of Minkowskian relativity. Historically, the style is linked to Minkowski's work; we will see later exactly how Minkowski used hyperbolic geometry to interpret the Lorentz transformation. Although "hyperbolic" and "non-Euclidean" geometry will refer here to geometry of constant negative curvature, the use of these terms varied in the period under consideration. The modern terminology of Minkowski space, or more generally, of pseudo-Euclidean spaces, had yet to enter the vocabulary of most mathematicians and physicists. Different writers described Minkowski's four-dimensional geometry as either "Euclidean," "non-Euclidean," or even as "hyperbolic."

The non-Euclidean style gave rise to a four-dimensional vector calculus like the space-time formalism, but one involving only real coordinates. The difference between the two formalisms hinges upon the treatment of the time coordinate t . In the space-time formalism, the temporal coordinate u is imaginary, $u \equiv ict$, where c is the universal light constant, and $i = \sqrt{-1}$. Imaginary coordinates are alien to the non-Euclidean calculus, which employs a different substitution for the temporal coordinate, $\ell \equiv ct$.

A rough comparison of the relative standing among scientists of the space-time formalism and the non-Euclidean style may be made, based on a simple frequency count based on usage in published articles. Our bibliographic database is compiled from titles in Lecat and Lecat-Pierlot (1924), with supplementary references from Hentschel (1990) and our own research. It covers articles published on relativity theory from 1908 through 1915 in West-European languages, in 130 journals (566 articles) and numerous anthologies (63 articles), for a total of 629 titles. We consider as an element of the set of relativist writings any publication in which the term "relativity" is invoked. In addition to this linguistic token, we seek a second, hybrid token, enlarging the relativist set with publications dealing with the Lorentz transformations in either a formal or a discursive fashion. For the sake of simplicity, we exclude from consideration all articles with titles invoking gravitation.⁷

Figure 1 compares the quantitative evolution of articles employing the space-time formalism and the non-Euclidean style, from 1908, when Minkowski's fundamental paper appeared, until 1916. Totalling the numbers of articles for this period, we find that space-time articles outnumber non-Euclidean publications by four to one.⁸ Articles featuring one or both of the approaches (144 titles by 62 authors) account for about one-fifth of all articles published on relativity between 1908 and 1916.

It is instructive to compare the publication details relating to our categories. For example, although a quarter of all relativist publications used the space-time formalism, they appeared

⁷Additional selection criteria are outlined in Walter (1996, chap. 4). For further statistics on the disciplinary structure of publications on relativity see Walter (1999, § 3).

⁸There are thirty non-Euclidean titles (including 7 reprints), as opposed to 117 space-time titles (with 19 reprints and 5 translations). Three articles employ both hyperbolic geometry and the space-time formalism, but the overlap between our categories is insignificant with respect to theoretical practice, since two of the three are review articles. In Figure 1, these three titles are represented in both categories.

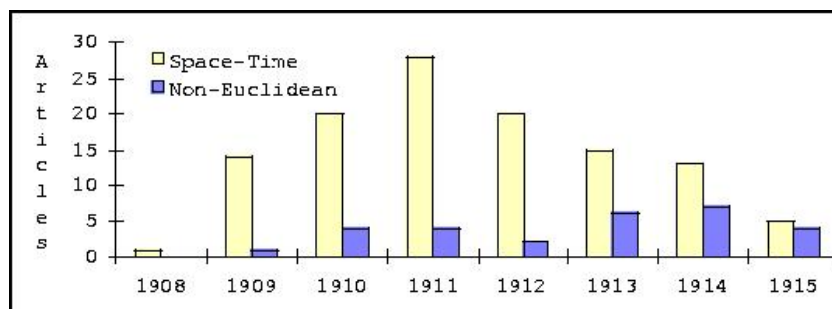


FIG 1. The space-time formalism and the non-Euclidean style ($N = 144$).

in only thirty-four journals. Physics journals, which publish almost half of all relativist articles (259 of 566), also carry half the space-time articles (51 of 100), and roughly a third of the non-Euclidean articles. By way of comparison, mathematics journals account for a tenth of all relativist articles, a fifth of the space-time articles, and a fifth of the non-Euclidean interpretations of relativity theory. The publishing organs of scientific academies and institutions account for most of the remaining articles in these categories. It is clear from these figures that articles employing the non-Euclidean style or the space-time formalisms were not excluded from physics journals in a systematic fashion.⁹

A more detailed image of the disciplinary structure of publications in this domain may be formed by correlating articles to the author's professional affiliation. The criterion for such affiliation in this instance is institutional: for the purposes of our study, disciplinary membership is determined by the title of the chair on which the writer depended. For non-titular university instructors, we determine affiliation by the position title, while for independent scholars, we use the dissertation advisor's discipline.

With these conventions, we find that our mathematicians are responsible for slightly more than a quarter of the relativist articles, including two-fifths of all space-time articles, and all but three of the thirty articles employing the non-Euclidean style. The remaining twenty-seven non-Euclidean articles come from a group of eleven mathematicians. Our physicists, by comparison, write nearly three-fifths of the space-time articles, and two-thirds of all relativist articles. Together, mathematicians and physicists account for over nine-tenths of the relativist articles from 1908 to 1916.

4 Minkowski's use of non-Euclidean geometry

The Göttingen professor of mathematics Hermann Minkowski delivered one of the first exposés of his views of the principle of relativity in November, 1907, before the assembled members of the Göttingen Mathematical Society. Rarely mentioned in the secondary literature, this lecture is nonetheless of particular interest, because it represents the only substantial statement of Minkowski's thoughts upon the principle of relativity before his discovery of the notion of proper time (*Eigenzeit*), with which he eventually elaborated the structure of space-time in terms of intersections of four-dimensional point trajectories (or "world-lines") and a Lorentz-covariant mechanics (Walter 1996, 101). Although one of the two surviving typescripts of the

⁹The idea of diverting to mathematics journals those submissions explaining physical phenomena in terms of non-Euclidean geometry was entertained in 1917 by an editor of the physics journal *Annalen der Physik* (Jungnickel and McCormmach 1986, vol. 2, 333).

lecture bears several annotations in Minkowski's hand, it is not clear that the text was intended for publication. The annotated typescript is our principle documentary source in this section.¹⁰

"The world in space and time," Minkowski claimed in his opening remarks, "is, in a certain sense, a four-dimensional non-Euclidean manifold."¹¹ His lecture would address the laws of physics, but he began by pointing out a "purely mathematical relation," concerning the differential equations used by the Dutch theorist H. A. Lorentz (1853–1928) as the foundation of his successful theory of electrons. These equations, Minkowski observed, were obviously independent of the particular choice of Cartesian axes in space, and yet they possessed a further symmetry, one not apparent in the notation ordinarily used for their expression. Minkowski then laid out the basis for a new system of notation, which he said had to do with the quadratic form

$$x^2 + y^2 + z^2 - c^2t^2,$$

where c stands for the velocity of propagation of light in empty space. Physical laws were to be expressed with respect to a four-dimensional manifold with coordinates x_1, x_2, x_3, x_4 , where ordinary Cartesian coordinates x, y , and z , went over into the first three, and the fourth was defined to be an imaginary time coordinate, $x_4 \equiv it$. When the units are chosen such that $c = 1$, Minkowski remarked, the above quadratic expression passes over to the form

$$x_1^2 + x_2^2 + x_3^2 + x_4^2.$$

Implicitly, Minkowski took as his formal starting point the final section of Poincaré's memoir on the dynamics of the electron (Poincaré 1906, § 8). He acknowledged the French mathematician's use of an imaginary temporal coordinate in the final section of his lecture, albeit in a rather oblique fashion, when he observed that Poincaré's search for a Lorentz-covariant law of gravitation involved the consideration of Lorentz-group invariants (Minkowski, "Das Relativitätsprinzip", 16, undated typescript, Cod. Ms. Math. Arch. 60: 3, Niedersächsische Staats- und Universitätsbibliothek, Göttingen). Poincaré used three real space coordinates and one imaginary temporal coordinate to define four-dimensional vectors for position, velocity, force, and force density. Did Minkowski then consider Poincaré to have anticipated his planned reformulation of the laws of physics in four dimensions?

As mentioned above, Minkowski described his contribution in terms of a notational improvement which revealed the symmetry shared by a certain quadratic form and the Maxwell-Lorentz electromagnetic field equations. Indeed, Minkowski distinguished his work on the principle of relativity from that of Einstein, Poincaré and Max Planck on precisely this basis:

Here I will bring to the notation from the beginning that symmetry, whereby the form of the equations, as I believe, really becomes extremely transparent. This is something brought out by none of the previously-mentioned authors, not even by Poincaré himself.¹² (Minkowski, "Das Relativitätsprinzip", 3, undated typescript,

¹⁰Six years after Minkowski's death, Arnold Sommerfeld published a document he claimed was the text of Minkowski's lecture (Minkowski 1915a, 1915b), but which differs substantially from the archival version. For discussions of the discrepancies, see Galison (1979) and Walter (1999, § 2.2).

¹¹"Es handelt sich, so kurz wie möglich ausgedrückt, genaueres werde ich alsbald ausführen, darum, dass die Welt in Raum und Zeit in gewissem Sinne eine vierdimensionale Nicht-Euklidische Mannigfaltigkeit ist" (Minkowski, "Das Relativitätsprinzip", 1, undated typescript, Cod. Ms. Math. Arch. 60: 3, Niedersächsische Staats- und Universitätsbibliothek, Göttingen).

¹²"Ich will hier, was übrigens bei keinem der genannten Autoren, selbst nicht bei Poincaré, geschehen ist, jene Symmetrie von vornherein zur Darstellung bringen, wodurch in der Tat die Form der Gleichungen wie ich meine äusserst durchsichtig wird." The phrase "wie ich meine" is an annotation in Minkowski's hand. Peter Galison first pointed out this key passage, but rendered it quite differently (Galison 1979, 104).

Cod. Ms. Math. Arch. 60: 3, Niedersächsische Staats- und Universitätsbibliothek, Göttingen)

In other words, Minkowski presented his main result as a notational one in which the Lorentz-covariance of the electromagnetic equations appeared as never before. He insisted here upon the fact that Poincaré did not write the Maxwell-Lorentz equations in four-dimensional terms.

This was hardly an oversight on Poincaré's part. The French scientist did not propose a four-dimensional vector calculus for general use, nor had he any intention of developing such a calculus for physics. While Poincaré recognized the feasibility of a translation of physics into the language of four-dimensional geometry, he said this would be "very difficult and produce few benefits." In this sense, he felt a four-dimensional vector calculus would be "much like Hertz's mechanics."¹³ Whether or not Minkowski was aware of Poincaré's view, it is quite clear that he did not share his opinion.

Once he had presented his central finding, Minkowski still had to show what a four-dimensional vector calculus has to do with a non-Euclidean manifold. When he reached the part of his lecture dealing with mechanics, Minkowski explained himself in the following way. The tip of a four-dimensional velocity vector w_1, w_2, w_3, w_4 , Minkowski stipulated,

is always a point on the surface

$$w_1^2 + w_2^2 + w_3^2 + w_4^2 = -1 \quad (1)$$

or, if you wish, on

$$t^2 - x^2 - y^2 - z^2 = 1, \quad (2)$$

and represents at the same time the four-dimensional vector from the origin to this point, and this also corresponds to null velocity, to rest, a genuine vector of this sort. Non-Euclidean geometry, of which I spoke earlier in an imprecise fashion, now unfolds for these velocity vectors.¹⁴ (Minkowski, "Das Relativitätsprinzip", 7, undated typescript, Cod. Ms. Math. Arch. 60: 3, Niedersächsische Staats- und Universitätsbibliothek, Göttingen)

Many mathematicians in Minkowski's audience probably recognized in (1) the equation of a pseudo-hypersphere of unit imaginary radius, and in (2) its real counterpart, the two-sheeted unit hyperboloid. Formally equivalent, both hypersurfaces provide a basis for a well-known model of non-Euclidean space of constant negative curvature, popularized by Helmholtz.¹⁵ Although Minkowski did not bother to unfold the geometry of velocity vectors, in the hypersurfaces (1) and (2), we have the premises of an explanation for Minkowski's description of the world as being—in a certain sense—a four-dimensional non-Euclidean manifold.

¹³"Il semble bien en effet qu'il serait possible de traduire notre physique dans le langage de la géométrie à quatre dimensions ; tenter cette traduction ce serait se donner beaucoup de mal pour peu de profit, et je me bornerai à citer la mécanique de Hertz où l'on voit quelque chose d'analogue" (Poincaré 1907, 15).

¹⁴"... so ist w_1, w_2, w_3, w_4 stets ein Punkt auf der Fläche $w_1^2 + w_2^2 + w_3^2 + w_4^2 = -1$ oder, wenn Sie wollen, auf $t^2 - x^2 - y^2 - z^2 = 1$, und repräsentiert zugleich den vierdimensionalen Vektor vom Nullpunkt nach diesem Punkte; und es entspricht auch der Geschwindigkeit Null, der Ruhe, ein wirklicher derartiger Vektor. Die Nicht-euklidische Geometrie, von der ich schon unbestimmt sprach, entwickelt sich nun für diese Geschwindigkeitsvektoren." Only equation (2) was numbered by Minkowski; we number equation (1) for clarity. While he referred to equations (1) and (2) as surfaces (*Flächen*), marginal annotations indicate that Minkowski considered three alternatives: world-surface, world-mirror and cosmograph (*Weltfläche, Weltspiegel, Kosmograph*); see Galison (1979, 116).

¹⁵See the appendix to Helmholtz's 1870 lecture on the origin and significance of the geometric axioms (Helmholtz 1884, vol. 2, 31). The model is also described in Clebsch & Lindemann (1891, 524).

The conjugate diameters of the hyperboloid (2), Minkowski went on to explain, give rise to a geometric image of the Lorentz transformation. Any point on (2) can be taken to lie on the t -diameter, and this change of axes corresponds to an orthogonal transformation of both the time and space coordinates which, as Minkowski observed, is a Lorentz transformation. Thus the three-dimensional hyperboloid (2) embedded in Minkowski's four-dimensional space affords an interpretation of the Lorentz transformation.

Over the years, Minkowski's terminology has generated significant confusion among commentators. In one sense, it appropriately underlined both the four-dimensionality of Minkowski's planned calculus, and the hyperbolic geometry of velocity vectors. Yet the label is flawed, for the following reason: although both the pseudo-hypersphere (1) and the two-sheeted unit hyperboloid (2) may be considered models of non-Euclidean space, neither one constitutes a four-dimensional manifold. Minkowski was surely aware of this ambiguity when he maintained that the label was only true "in a certain sense."

In any case, Minkowski never again referred to a manifold as both four-dimensional and non-Euclidean. Along with the problematic label, the geometric interpretation of velocity vectors likewise vanishes from view in Minkowski's subsequent writings. Felix Klein, for one, regretted the change; in his opinion, Minkowski later hid from view his "innermost mathematical, especially invariant-theoretical thoughts" on the theory of relativity (Klein 1927, 75).

Six months after his lecture to the Göttingen Mathematical Society, Minkowski published his first essay on the principle of relativity. Entitled "The Basic Equations of Electromagnetic Processes in Moving Bodies," it presented a new theory of the electrodynamics of moving media, incorporating formal insights of the relativity theories introduced earlier by Einstein, Poincaré and Planck. For example, it took over the fact that the Lorentz transformations form a group, and that Maxwell's equations are covariant under this group. Minkowski also shared Poincaré's view of the Lorentz transformation as a rotation in a four-dimensional space with one imaginary coordinate, and his five four-vector expressions.

These insights Minkowski developed and presented in an original, four-dimensional approach to the Maxwell-Lorentz vacuum equations, the electrodynamics of moving media, and in an appendix, Lorentz-covariant mechanics. In the sophistication of its mathematical expression, Minkowski's paper rivalled that of Poincaré, acknowledged by many to be the world's leading mathematician. One aspect of the principle of relativity, according to Minkowski, made it an excellent object of mathematical study:

To the mathematician, accustomed to contemplating multi-dimensional manifolds, and also to the conceptual layout of the so-called non-Euclidean geometry, adapting the concept of time to the application of Lorentz transformations can give rise to no real difficulty.¹⁶ (Minkowski 1908, 69–70)

Understanding the concept of time in the theory of relativity, in other words, represented no challenge for mathematicians because of their experience in handling similar concepts from n -dimensional and non-Euclidean geometry. This is not the only time Minkowski encouraged mathematicians to study the principle of relativity in virtue of its mathematical or geometrical form. His Cologne lecture was to go even further in this direction, by suggesting that the essence of the principle of relativity was purely mathematical (Walter 1999, §2.1). In the passage quoted above, Minkowski's claim is a less general one, to the effect that mathematicians'

¹⁶"Dem Mathematiker, der an Betrachtungen über mehrdimensionale Mannigfaltigkeiten und andererseits an die Begriffsbildungen der sogenannten nicht-Euklidischen Geometrie gewöhnt ist, kann es keine wesentliche Schwierigkeit bereiten, den Begriff der Zeit an die Verwendung der Lorentz-Transformationen zu adaptieren."

familiarity with non-Euclidean geometry would allow them to handle the concept of time in the Lorentz transformations. Yet apart from this disciplinary aside, the subject of non-Euclidean geometry is conspicuously absent from all that Minkowski published on relativity theory.

On the other hand, Minkowski retained the geometric interpretation of the Lorentz transformations that had accompanied the now-banished non-Euclidean interpretation of velocity vectors. In doing so, he elaborated the notion of velocity as a rotation in four-dimensional space. He introduced a formula for the frame velocity q in terms of the tangent of an imaginary angle $i\psi$, such that

$$q = -i \tan i\psi = (e^\psi - e^{-\psi}) / (e^\psi + e^{-\psi}).$$

Minkowski could very well have expressed frame velocity in the equivalent form $q = \tanh \psi$, where the angle of rotation is real instead of imaginary, and all four space-time coordinates are real. He did not do so, but used the imaginary rotation angle $i\psi$ to express the special Lorentz transformation in the trigonometric form:

$$x'_1 = x_1, \quad x'_2 = x_2, \quad x'_3 = x_3 \cos i\psi + x_4 \sin i\psi, \quad x'_4 = -x_3 \sin i\psi + x_4 \cos i\psi.$$

The use of circular functions here underscores the fact that a special Lorentz transformation is equivalent to a rotation in the (x_3x_4) -plane. Likewise, by expressing velocity in terms of an imaginary rotation, Minkowski may have betrayed his knowledge of the formal connection between the composition of Lorentz transformations and relative velocity addition, remarked earlier by Einstein on different grounds (Einstein 1905, §5). However, Minkowski neither mentioned the law of velocity addition, nor expressed it in formal terms.

Minkowski's preference for circular functions may be understood in relation to his project to express the laws of physics in four-dimensional terms. Four-dimensional vector algebra is a natural extension of the ordinary vector analysis of Euclidean space when the time coordinate is multiplied by $\sqrt{-1}$. Expressing the Lorentz transformation as a hyperbolic rotation would have obscured the connection for physicists.

Mathematicians, on the other hand, had little use for vector analysis, and were unlikely to be put off by the use of hyperbolic functions. Judging from his correspondence, Minkowski was not shy of hyperbolic geometry. In a postcard sent to his former teacher and friend, the Zürich mathematician Adolf Hurwitz (1859–1919), Minkowski described the “quintessence” of his relativity paper as the “Principle of the Hyperbolic World.”¹⁷

The link to ordinary Euclidean space from four-dimensional space-time was one of the themes Minkowski stressed in his lecture to the scientists assembled in Cologne for the annual meeting of the German Association in September, 1908. Under the new space-time view, Minkowski announced, “Three-dimensional geometry becomes a chapter of four-dimensional physics.”¹⁸ In the same triumphant spirit, Minkowski suggested that his new four-dimensional understanding of the laws of physics deserved its own label. The “Principle of the Hyperbolic World” that he had tried on Hurwitz was shelved in favor of the more ecumenical “Postulate of the Absolute World” (Minkowski 1909, 82). Although Minkowski explained this to mean that only the four-dimensional world in space and time is given by phenomena (Minkowski 1909, 82), one suspects an inside joke with Hurwitz, since in the German mathematical community, hyperbolic geometry was sometimes referred to as absolute geometry.

Although the lecture on “Space and Time” was read to the mathematics section of the Cologne meeting, it recapitulated a selection of Minkowski's results so that they could be

¹⁷H. Minkowski to A. Hurwitz, May 5, 1908, Cod. Ms. Math. Arch. 78: 212, Niedersächsische Staats- und Universitätsbibliothek.

¹⁸“Die dreidimensionale Geometrie wird ein Kapitel der vierdimensionalen Physik” (Minkowski 1909, 79).

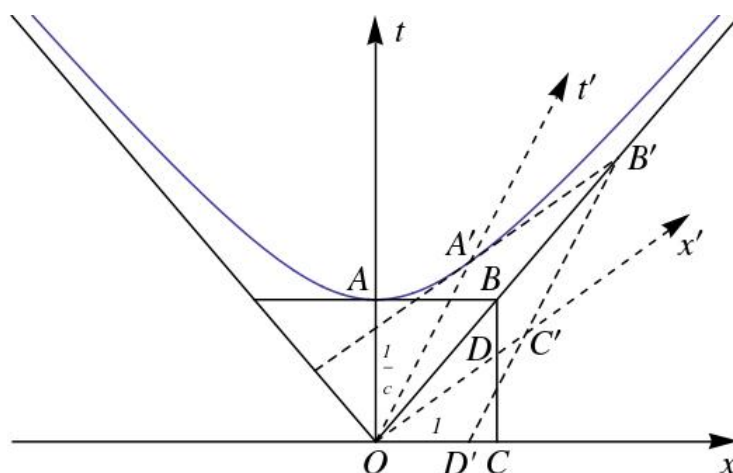


FIG 2. Minkowski's space-time diagram, after Minkowski (1909, 77).

understood by those with little mathematical training. Notably, it displayed the fundamental four-vectors of relativistic mechanics, while neglecting the finer points of his matrix calculus. Likewise, the above-mentioned interpretation of Lorentz transformations with respect to a real hyperboloid (2) resurfaced in a two-dimensional version lending itself to graphical illustration. Minkowski's space-time diagram (see Figure 2) refers to an invariant hyperbola in the (xt) -plane, $t^2 - x^2 = 1$, which is just equation (2) without the y and z coordinates. The diagonals describing the equations $x = t$ and $x = -t$ correspond here to the paths of light rays in empty space that pass through the origin of the coordinate system; in four-dimensional space-time these rays form an invariant hypercone. While Minkowski did not bother to show this himself, from the geometrical relations of the diagram one may derive the special Lorentz transformation.¹⁹

From a retrospective standpoint, Minkowski's Cologne lecture was instrumental in transforming the principle of relativity from a peculiar problem in electron dynamics into the most celebrated discovery in contemporary theoretical physics. Several contemporary observers saw in Minkowski's formalism a new approach to the principle of relativity, yet one that shared with the theories of Poincaré and Einstein the requirement of covariance of the laws of physics with respect to the Lorentz transformations. Perhaps most importantly in this respect, Minkowski's four fundamental equations of electromagnetism were understood by Max Laue and others to be a summary of atomistic electrodynamics in its entirety. Laue observed at the same time that the proof that these equations satisfy the principle of relativity and the requirements of conservation of energy and momentum resides in their form alone.²⁰

In the period immediately following the Cologne lecture there was a significant upswing in the number of publications mentioning the principle of relativity. By the end of 1909, Minkowski and five other theorists had published a total of fourteen articles using the space-time formalism.²¹ Also important in the diffusion of the space-time formalism was Max Laue's relativity textbook (Laue 1911), which extended Arnold Sommerfeld's four-dimensional vector algebra (Sommerfeld 1910a, 1910b) in a systematic and elegant approach to relativity theory.

¹⁹For examples, see Laue (1911, 47), or the appendix to Walter (1999).

²⁰Laue 1911, 88. Minkowski's results are compared to those of Einstein by Holton (1965), Pyenson (1985, chap. 4), Galison (1979), and Walter (1999, §2.5). For succinct comparisons with the work of Poincaré, see Cuvaj (1968) and Walter (1999, §2.2).

²¹Ten of the fourteen titles were written by either Minkowski or Max Born, yet of these ten, six are reprints or translations. On responses to the Cologne lecture, see Walter (1999).

Through the efforts of these physicists, and of others like Max Abraham (1910) and Gilbert N. Lewis (1910a), Minkowski's matrix calculus was transformed first into a convenient four-dimensional vector analysis, and eventually into a tensor calculus.²² By 1911, four-vector and six-vector operations featured prominently in the pages of the leading physics journal in Germany, the *Annalen der Physik*. Out of the nine theoretical papers on relativity theory published in the *Annalen* that year, eight applied the space-time formalism. By 1912, this formalism had become the standard for advanced research in relativity.²³

While the older coordinate and vector approaches to relativity were effectively displaced from the *Annalen* by the space-time formalism, they did not disappear from use by any means. The second edition of Laue's textbook (Laue 1913), for example, which relies heavily upon the new space-time formalism, includes an appendix on ordinary vector analysis. Of the four textbooks available on the subject of relativity by 1914, three employ a mix of three and four-dimensional entities (Laue 1911; Silberstein 1914; Cunningham 1914). Ludwig Silberstein's text is perhaps an extreme version of this eclectic approach to notation. In addition to Cartesian coordinates, ordinary space vectors, and space-time vectors, Silberstein introduced Cayley matrices and quaternions, neither of which were to gain a significant following, however.

The exception to the rule of using notational shortcuts is Max B. Weinstein's treatise (Weinstein 1913). A translator of Maxwell and Kelvin, the Berlin philosopher-physicist Weinstein (1852–1918) was like Minkowski a Russian immigrant. While Weinstein dedicated his treatise to the memory of Minkowski, he was still critical of his mathematical technique, finding this (p. vi) “unspeakably difficult to understand.” The difficulty undoubtedly stemmed in part from his lack of confidence in the complex quantities upon which Minkowski built his calculus. Weinstein deplored formulae with imaginary terms because they defied visualization; for him they were *unvorstellbar*.²⁴ The four-dimensional operators simply had to go, so Weinstein expressed Minkowski's theory in terms of either ordinary vectors or orthogonal Cartesian coordinates, usually writing out every component in full. In doing so, Weinstein claimed (p. vi) to have clarified Minkowski's “brilliant achievement,” and “place[d] it on a human level.”

Two of Weinstein's readers disagreed with him on this count. One of the critics was Roberto Marcolongo (1862–1943), a professor of rational mechanics in Naples, who had his own three-dimensional vector calculus to promote (Marcolongo 1914, 452, note 14). Along with Marcolongo, Max Born (1882–1970), the Göttingen *Privatdozent* in theoretical physics and Minkowski's devoted disciple, decried Weinstein's “heaps of formulæ” (*Formelhaufen*, Born (1914). The strident tone of Born's review reflects his strong preference—shared by many theorists—for vector and tensor formulations of the laws of physics.

From the above survey of publications in the *Annalen der Physik*, it appears that the space-time formalism had gained the confidence of leading theorists by 1912. The vector and tensor reformulation of Minkowski's calculus played an essential role in this respect. Likewise, the advocacy of respected theorists like Sommerfeld and Abraham no doubt encouraged others to try out the space-time formalism. Perhaps more decisive in swaying opinion than either the vectorial reformulation or the commanding example of leading theorists was the apparent superiority of the space-time formalism over ordinary vector analysis or Cartesian coordinate methods.

In order to understand better how the space-time formalism came to be the dominant style

²²On the development of vector and tensor methods, see Crowe (1967) and Reich (1994).

²³On the influence of the theory of relativity on the mathematical sophistication of contributions to physics journals in Germany, see Jungnickel & McCormmach (1986, vol. 2, 313).

²⁴Weinstein 1913, 307, note 1. As Pyenson observes (1985, 150), Weinstein was also wary of the use of differential equations in physics.

in theoretical investigations concerning the principle of relativity, we must examine the content of the relevant publications of the period. Through a close reading of the latter, we can try to discern those features in the early applications of the space-time formalism which either attracted, repelled, or left scientists indifferent. Within this interpretational framework, let us review a selection of scientific responses to a peculiar, latent feature of Minkowskian relativity: non-Euclidean geometry.

5 Non-Euclidean readings of Minkowski

Paul Mansion (1844-1919) was a Belgian mathematician, editor of the journal *Mathesis* and author of over sixty articles on non-Euclidean geometry. In a review of Minkowski's Cologne lecture, Mansion shared his impression that, as far as he could tell, "consciously or unconsciously (Minkowski) applies non-Euclidean geometry to physics."²⁵ For Mansion, moreover, this physical application of non-Euclidean geometry "explains rather easily" both Lorentz's "paradoxical proposition" concerning the longitudinal contraction of bodies in motion, and Einstein's "complementary remark" on the equivalence of inertial frames of reference (Mansion 1909, 245).

Mansion's review suggests that there was no real difficulty in considering Minkowski's work as the latest entrant in a fashionable trend of studies of non-Euclidean mechanics. Reviewing the bibliography of non-Euclidean and n -dimensional geometry mentioned above, the British mathematician G. B. Mathews (1861–1922) felt it was "not unlikely" that the topics covered would be "of wholly unexpected importance in the applications of mathematics to physics." Observing how the volumes of Minkowski's collected memoirs formed the final entry of Sommerville's list, the reviewer asked if anything could be "more suggestive" (Mathews 1912).

Minkowski's suppression of all but the most vague reference to non-Euclidean geometry may well have made his relativist publications more acceptable to physicists, but it did not shield them from criticism on this ground. Less than two weeks after Minkowski's theory of the electrodynamics of moving media appeared in print, Einstein wrote to his wife with great news: on the basis of Jakob Laub's calculations, he had found an error concerning the definition of ponderomotive force density.²⁶ Together, Einstein and Laub came up with an alternative definition.²⁷ In a companion paper, they set about rederiving Minkowski's equations using ordinary vector analysis, because they felt Minkowski's four-dimensional formalism asked too much of the reader.²⁸

Einstein and Laub were not favorably impressed by Minkowski's four-dimensional calcu-

²⁵"Autant que nous pouvons saisir l'exposition trop condensée de Minkowski, il nous semble qu'au fond, il applique consciemment ou inconsciemment la géométrie non euclidienne à la physique." Mansion 1909, 245.

²⁶A. Einstein to Mileva Einstein-Marić, April 17, 1908, Einstein (1993b, Doc. 96). Like Einstein, Jakob Laub (1882–1962) was a former student of Minkowski; he established the notes for Minkowski's lectures on mechanics in Göttingen, and participated in the electron theory seminar led by Minkowski and David Hilbert in the summer semester of 1905. Laub then undertook an experimental investigation of cathode ray emission under Willy Wien's direction in Würzburg (Pyenson 1985, 220).

²⁷Einstein & Laub 1908a. Einstein devised arguments in defense of their formula in 1910, but lost interest in it some time later. In a letter to Walter Dällenbach in 1918, Einstein candidly remarked that it had been known for some time that the expression he and Laub devised was false (Fölsing 1993, 276). For a succinct account of the issues involved, and additional references, see the editorial note in Einstein's *Collected Papers* (Stachel 1989, 503).

²⁸Einstein & Laub 1908b, 532. Although the technical intricacy of Cayley matrix calculus was disputed by Max Born (1909a, 7) and Felix Klein (1927, 75), very few theorists adopted it.

lus, and believed his theory of the electrodynamics of moving media to contain at least one incorrect formula. Naturally, Laub was curious to know what others thought about Minkowski's approach. He asked the Würzburg theoretical physicist Mathias Cantor (1861–1916) what he considered to be the “real physical meaning of time as a fourth spatial coordinate” in Minkowski's theory, without getting an answer. Recounting this episode in a letter to Einstein, Laub opined that Cantor “let himself be impressed by non-Euclidean geometry.”²⁹

In the spring of 1909, Max Planck (1858–1947), the leading spokesman for theoretical physics in Germany, delivered a series of eight lectures at Columbia University. In the last of these, he turned his attention to the principle of relativity. Planck lavished praise on Einstein for his modification of the concept of time:

It need scarcely be emphasized that this new view of the concept of time makes the most serious demands upon the capacity of abstraction and the imaginative power of the physicist. It surpasses in boldness everything achieved so far in speculative investigations of nature, and even in philosophical theories of knowledge: non-Euclidean geometry is child's play in comparison.³⁰ (Planck 1910a, 117)

Planck certainly meant to underline Einstein's accomplishment and its significant philosophical consequences. As John Heilbron observes, Planck was a key figure in securing acceptance of Einstein's work in Germany.³¹ The comparison of non-Euclidean geometry to child's play, however, was most likely a rejoinder to Minkowski's remark on time, according to which *mathematicians* were uniquely well-equipped to understand the notion of time in the Lorentz transformations.

Where Minkowski underlined the conceptual continuity of non-Euclidean geometry and the notion of time in relativity, Planck refused the analogy, and emphasized the revolutionary nature of Einstein's new insight.³² For Planck, however, there was at least an historical similarity between non-Euclidean geometry and relativity. The relativity revolution was similar to that engendered by the introduction of non-Euclidean geometry: after a violent struggle, Planck recalled, the *Modernisten* finally won general acceptance of their doctrine (Planck 1910b, 42–43).

In his address to the German Association in September, 1910, Planck acknowledged that progress in solving the abstract problems connected with the principle of relativity was largely the work of mathematicians. The advantage of mathematicians, Planck noted (Planck 1910b, 42), rested in the fact that the “standard mathematical methods” of relativity were “entirely the same as those developed in four-dimensional geometry.” Thus for Planck, the space-time formalism had already become the standard for theoretical investigations of the principle of relativity.

²⁹Klein 1993b, Doc. 101; Pyenson 1985, 225.

³⁰“Es braucht kaum hervorgehoben zu werden, daß diese neue Auffassung des Zeitbegriffs and die Abstraktionsfähigkeit und an die Einbildungskraft des Physikers die allerhöchsten Anforderungen stellt. Sie übertrifft an Kühnheit wohl alles, was bisher in der spekulativen Naturforschung, ja in der philosophischen Erkenntnistheorie geleistet wurde; die nichteuclidische Geometrie ist Kinderspiel dagegen.”

³¹Heilbron 1986, 28. Planck's flattering characterizations of Einstein's work (he once compared Einstein to Copernicus) also did the young man's career a great service. Thus in 1910, a selection committee at the Germany University in Prague cited the remarks quoted above in favor of Einstein's appointment to the chair of theoretical physics (Illy 1979, 76).

³²After Thomas Kuhn (1962), several historians have explored the theme of revolution in relation to the reception of the special theory of relativity (Illy 1981; Pyenson 1987; I. B. Cohen 1985). In a different vein, Norton Wise (1983) discussed the historicist movement in late nineteenth-century physics.

Planck's coeditor at the *Annalen der Physik*, Willy Wien (1864–1928), reiterated the contrast between non-Euclidean geometry and physics in his review of Einstein's and Minkowski's views of space and time. Wien portrayed Einstein's theory of relativity as an induction from results in experimental physics; here, according to Wien (1909, 30), there was "no direct point of contact with non-Euclidean geometry." Minkowski's theory, on the other hand, was associated in Wien's lecture with a different line of development: the abstract, speculative theories of geometry invented by mathematicians from Carl Friedrich Gauss to David Hilbert.

Wien admitted there was something "extraordinarily compelling" about Minkowski's view. The whole Minkowskian system, he said, "evokes the conviction that the facts would have to join it as a fully internal consequence." As an example of this, he mentioned Minkowski's four equations of motion, the fourth of which is also the law of energy conservation (Minkowski 1909, 85). Wien nonetheless distanced himself from the formal principles embodied in Minkowski's contribution to relativity when he recalled that the physicist's credo was not aesthetics but experiment. "For the physicist," Wien concluded, "Nature alone must make the final decision."³³

Certain experimental consequences of Minkowski's theory of the electrodynamics of moving media awaited experimental investigation, which may be what motivated Wien's conclusion. At the time, new experiments designed to test the predictions of relativity theories were in scarce supply. Among the best-known of these were the electron-deflection experiments run by Walter Kaufmann and Alfred Bucherer, which gave conflicting results and elicited significant controversy.³⁴ Since Einstein's theory of relativity and Minkowski's space-time theory were generally understood to stand or fall on the same empirical base, the comparison between them could proceed only on either formal or methodological grounds.

In summary, certain mathematicians and physicists cast Minkowski's work in a tradition of research on non-Euclidean geometry. For the mathematicians Mansion and Mathews, relativity theory was ripe for study and development by geometers. The physicists Planck and Wien, on the other hand, denied any link between non-Euclidean geometry and Einstein's theory of relativity. But like Mansion and Mathews, Wien considered Minkowski's theory to belong to a tradition of speculative research in non-Euclidean geometry, strongly associated with Göttingen mathematicians.

The responses to Minkowski's theory reviewed in this section suggest that the value accorded to Minkowski's geometric approach to physics depended on professional affiliation. Yet opinion of Minkowski's work was certainly not divided along disciplinary lines in an absolute sense. Not all relativist mathematicians admired Minkowski's four-dimensional physics; Henri Poincaré and Ebenezer Cunningham, for instance, both expressed a preference for Lorentz's approach to the electrodynamics of moving bodies (Poincaré 1912, 170; Cunningham 1912, 126). Likewise, several physicists, in particular those who had ties to Göttingen (Max Abraham, Max Born, Arnold Sommerfeld), were convinced that the space-time formalism was superior in some ways to the older methods.

³³"Es lässt sich nicht leugnen, dass diese veränderte Auffassung von Raum und Zeit etwas ungemein Zwingendes hat und dass das ganze System voll innerer Konsequenz die Überzeugung wachruft die Tatsachen müssten sich ihm fügen. Für den Physiker kommen aber ebensowenig ästhetische Momente in Betracht wie es früher mit teleologischen der Fall war. Für ihn hat allein die Natur die Entscheidung zu treffen." Wien (1909, 30). For a different reading of Wien's lecture see Pyenson (1985, 145).

³⁴For details on these experiments, see Miller (1981).

6 A convenient guide: Arnold Sommerfeld on velocity composition

Minkowski's visually-intuitive description of space-time geometry fired the imagination of many a scientist, but in its first year of existence, his algebraic formalism made few inroads into theoretical practice. During this period, his former assistant Max Born (1882–1970) was the only one to apply the formalism (Born 1909b).

The first ones to comment in print on Minkowski's theory, Einstein and Laub considered its mathematical form an obstacle to comprehension, as mentioned above, and went on to rederive its basic equations in the more familiar notation of vector calculus. Similarly, in his early work on space-time mechanics, Philipp Frank (1884–1966) made no use of the four-dimensional apparatus in which the theory was originally couched, relying instead upon ordinary vector methods in all his calculations.³⁵ These bright young theorists felt more comfortable with ordinary vector analysis than with Minkowski's matrix formalism, yet they were still able to understand the theory, and to express it in more familiar terms.

When read by non-theorists, on the other hand, Minkowski's publications triggered attacks of mathematics anxiety. Even the watered-down version of the space-time theory presented in Minkowski's Cologne lecture repelled some physicists. For instance, Willy Wien's cousin Max (1866–1938), a physicist at Danzig Polytechnic, confided to his friend Arnold Sommerfeld that reading Minkowski gave him vertigo:

Sommer maintains that [Minkowski's] speech in Cologne was simply grand; when reading it, however, I always get a slight brain-shiver, now (that) space and time appear conglomerated together in a gray, miserable chaos.³⁶ (Max Wien to Arnold Sommerfeld, February 16, 1909, Benz 1975, 71)

Thus unlike his mathematical colleague Julius Sommer (1871–1943), Max Wien was not inspired by the idea of referring the laws of physics to a space-time manifold. And while Wien appeared to admit the validity of Minkowskian relativity, his willingness to develop the theory and investigate its experimental consequences was undoubtedly compromised by its perceived abstraction.

At the September, 1909, meeting of the German Association of Natural Scientists and Physicians in Salzburg, Arnold Sommerfeld (1868–1951) attempted to spark physicists' interest in Minkowski's formalism. A former assistant to Felix Klein, Sommerfeld succeeded Ludwig Boltzmann in the Munich chair of theoretical physics in 1906.³⁷ At first skeptical of Einstein's theory, Sommerfeld found Minkowski's space-time theory highly persuasive, and following Minkowski's death, became its most distinguished advocate in physics (Walter 1999, §3.1).

In his Salzburg talk, Sommerfeld insisted upon the practical advantage in problem solving offered by the space-time view:

Minkowski's profound space-time view not only facilitates the general construction of the relative-theory in (a) systematic way, but also proves successful as a

³⁵Frank 1908, 897. In March, 1909, Frank expressed the special Lorentz transformation and frame velocity with hyperbolic functions, thus inaugurating the non-Euclidean style (Frank 1909).

³⁶“Sommer behauptet, seine Rede in Köln sei einfach grossartig; ich kriege beim Lesen aber immer einen leisen Gehirntatterich, nur Raum und Zeit scheinen sich zu einem grauen, elenden Chaos zusammen zu ballen.”

³⁷Eckert & Pricha 1984; Jungnickel & McCormach 1986, vol. 2, 278.

convenient guide in specific problems.³⁸ (Sommerfeld 1909, 829)

As an example of the advantage of the Minkowskian approach, Sommerfeld selected the case of Einstein's "famous addition theorem," according to which velocity parallelograms do not close. This "somewhat strange" result, Sommerfeld suggested, became "completely clear" (*völlig durchsichtig*) when viewed from Minkowski's standpoint. From our review of Minkowski's theory, we recall how he introduced a formula for frame velocity in terms of the tangent of an imaginary angle $i\psi$, and expressed the special Lorentz transformation in trigonometric form. Sommerfeld borrowed the latter form of the transformation, writing ϕ instead of $i\psi$; this substitution underlined what Sommerfeld called the "analogy" between ordinary space rotations and space-time rotations. In analytic language, Sommerfeld added, this analogy was actually an identity.

With this formal basis, Sommerfeld derived Einstein's expressions for velocity composition for the two cases corresponding to the law's special and general form. In the special case of parallel velocities, Sommerfeld simply applied the standard formula for addition of tangents. By considering frame velocity in a trigonometric form, in other words, Sommerfeld showed that velocity composition for two systems in uniform, parallel motion amounts to summing tangents. Updating Sommerfeld's notation a little, we can write his expression for relative velocity:

$$\beta = \frac{1}{i} \tan(\phi_1 + \phi_2) = \frac{1}{i} \frac{\tan \phi_1 + \tan \phi_2}{1 - \tan \phi_1 \tan \phi_2} = \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2},$$

where $\beta = v/c$, and the subscripts correspond to two systems in uniform parallel motion. The formal concision and conceptual simplicity of Sommerfeld's derivation were widely appreciated; some years later even Einstein adopted the method (Miller 1981, 281, note 4).

For the more general case of two inertial systems moving in different directions, Sommerfeld interpreted the imaginary rotation angle ϕ as an arc of a great circle on a sphere of imaginary radius. Here the relative velocity of an arbitrary point with respect to any two systems of reference in uniform motion is found by constructing a triangle on the surface of the sphere, the sides of which follow from the cosine law (see Figure 3).³⁹

Sommerfeld did not mention non-Euclidean geometry in so many words, yet the surface of a hemisphere of imaginary radius was a well-known model of hyperbolic geometry, as mentioned earlier. Sommerfeld's spherical-trigonometric formulae employing an imaginary angle can be rewritten in terms of *real* hyperbolic trigonometry, a fact which was unlikely to have escaped him. In all likelihood, Sommerfeld wished to appeal to physicists' spatial intuition, as further witnessed by the three figures accompanying his article. Spherical trigonometry undoubtedly represented for Sommerfeld the clearest means of presenting his ideas to physicists.

Yet the artifice of an imaginary sphere was judged excessively abstract by one of Sommerfeld's readers, the mathematical physicist Ludwik Silberstein (1872–1948). Silberstein suggested that instead of an imaginary sphere, one could use the pseudosphere to study the properties of velocity composition. Such surfaces, he noted in his textbook on relativity, were found in many mathematical classrooms, and could render the subject accessible "even to all those who do not like to think of hyperbolic, and other non-Euclidean, spaces." When we recall that Silberstein's treatise was itself regarded by one reviewer as excessively mathematical, Sommerfeld's neglect of a more explicit use of non-Euclidean geometry appears fully

³⁸"Der einzige Zweck dieser kleinen Mitteilung war der, zu zeigen, daß die tief sinnige Raumzeit-Auffassung Minkowskis nicht nur in systematischer Hinsicht den allgemeinen Aufbau der Relativtheorie erleichtert, sondern sich auch bei speziellen Fragen als bequemer Führer bewährt."

³⁹For a detailed exposé of Sommerfeld's model, see Rosenfeld (1988, 270).

deplored the “purely analytic character” of Minkowski’s work (Robb 1913, 5). In a formal sense, at least, the employment of hyperbolic trigonometry in Robb’s optical geometry distinguished his work from that of the Minkowskians, just as it simplified his calculations, and fed his spatial intuition.

8 Vladimir Varičak’s non-Euclidean program

Vladimir Varičak (1865–1942) was a professor of mathematics at the University of Agram (now Zagreb, Croatia), and author of several studies of hyperbolic geometry. In Varičak’s hands, Sommerfeld’s trigonometry on an imaginary sphere became real hyperbolic trigonometry. The representation of velocity composition and of Lorentz transformations with respect to hyperbolic space formed the basis of Varičak’s program to approach the theory of relativity from the standpoint of non-Euclidean geometry.

More than any other mathematician, Varičak devoted himself to the development and promotion of the non-Euclidean style, unfolding Minkowski’s image of velocity-vector relations in hyperbolic space, and recapitulating a variety of results in terms of hyperbolic functions. The use of hyperbolic trigonometry was shown by Varičak to entail significant notational advantages. For example, he relayed the interpretation put forth by Herglotz and Klein of the Lorentz transformation as a displacement in hyperbolic space, and indicated simple expressions for proper time and the aberration of light in terms of a hyperbolic argument.⁴²

In recognition of his accomplishment, Varičak received an invitation to report on this new sub-branch of applied geometry to the German Society of Mathematicians at its annual meeting, in joint session with the German Association in Karlsruhe in 1911. Other speakers on relativity in the mathematics section included two well-known geometers: Josef Wellstein (1869–1919) of the University of Strasbourg, and Lothar Heffter (1862–1962), the newly-named professor of mathematics at the University of Freiburg; altogether some twenty-two mathematicians gave talks at this meeting.⁴³

The mathematicians were not the only ones interested in relativity, of course. In the physics section, Sommerfeld was asked by the German Physical Society to deliver a plenary lecture on the theory of relativity. He demurred, explaining that this could no longer be considered one of the current objects of research; relativity had become the “secure property of physics.”⁴⁴ In the two years following Sommerfeld’s initial promotional effort (see § 5), the outlook for the space-time formalism had improved considerably. By the end of 1911, as mentioned above, the space-time formalism had displaced ordinary vector calculus as the tool of choice for research in relativity.

What Sommerfeld chose to lecture on instead of relativity was the recent work related to Einstein’s energy quantum, including Sommerfeld’s own quantum theory. Sommerfeld considered Planck’s quantum of action to be the most promising basis for future work in this area, not least because it has the property of Lorentz invariance. The latter property he outlined in a special section on relativity, where he reviewed the fundamentals of the space-time formalism, and expressed action in terms of the four-dimensional line element.⁴⁵

Undoubtedly, not all those present in Karlsruhe found themselves in full agreement with Sommerfeld’s assessment of the research prospects in relativity theory. Varičak, for example,

⁴²Herglotz 1910; Klein 1910; Varičak 1912.

⁴³*L’Enseignement mathématique* **13** (1911), 514.)

⁴⁴Sommerfeld 1911, 1057.

⁴⁵Sommerfeld 1911, §8. For background on this paper, see Kuhn (1978, 226).

considered the theory of relativity to be a fertile domain for research; his own pace of publication in this domain did not let up for years. Sommerfeld and Varičák were both right in a way, since the number of articles published annually on relativity (excluding gravitational theories, see § 2) drops after 1911 for physicists, while for mathematicians there is no decline until the onset of the First World War (Walter 1999, § 3).

Varičák was well aware of a difference of opinion concerning the role of non-Euclidean geometry in relativity, as he contrasted Minkowski's view on this question to those of Planck and Wien. He did not claim that Planck's and Wien's pronouncements were ill-informed, but in the circumstances, this would have been superfluous. Wien, for one, had silently retracted his opinion (see § 4 above), by excising the offending passage of his 1909 lecture for reedition in Felix Auerbach and Rudolf Rothe's popular handbook, the *Taschenbuch für Mathematiker und Physiker*. The reputation of the non-Euclidean style was well enough established for Varičák to consider the earlier opinions of the editors of the *Annalen der Physik* as fully refuted.

In his review of opinion on the role of non-Euclidean geometry in relativity theory, Varičák neglected to mention the view of his most powerful critic, who happened to give the keynote address in physics that year. Two years earlier, just after Varičák's first exposé of the non-Euclidean style (Varičák 1910), Sommerfeld completed his signal work on the four-dimensional vector calculus for the *Annalen der Physik*. In a footnote to this work, Sommerfeld remarked that the geometrical relations he presented in terms of three real and one imaginary coordinate could be reinterpreted in terms of non-Euclidean geometry. The latter approach, Sommerfeld cautioned, could "hardly be recommended" (Sommerfeld 1910a, 752).

Equally omitted from Varičák's report was his explanation of the Lorentz-FitzGerald contraction (according to which all moving bodies shrink in their direction of motion with respect to the ether) as a psychological phenomenon. Earlier in the year, Einstein had contested his argument by maintaining the reality of the contraction.⁴⁶

Thus ignoring both Sommerfeld's dim view of his non-Euclidean program, and Einstein's correction of his interpretation of relativity theory, Varičák went on to demonstrate the formal simplicity afforded by hyperbolic functions in the theory of relativity. Such a remarkable fit between geometry and physics could not be fortuitous, so Varičák stated that after writing his first papers interpreting the formulae of relativity with non-Euclidean geometry, he changed his orientation, by assuming phenomenal space to be not Euclidean but hyperbolic, such that physical phenomena "pre-occur" in hyperbolic space (Varičák 1912, 105).

Varičák's radical ontological switch mimicked that of Minkowski, who argued in his Cologne lecture that the seat of physical reality is four-dimensional space-time, as mentioned earlier. It was hailed by two lesser-known figures: G. B. Halsted (1853–1922), a retired mathematician from Colorado (Halsted 1912, 597), and Paul Riebesell (1883–1950), a secondary-school teacher in Hamburg trained in mathematical physics (Riebesell 1916, 99). Others ignored Varičák's conjecture.

The change to a non-Euclidean perspective was conservative in one sense, for with non-Euclidean terminology, Varičák argued,

... the formulæ of the theory of relativity are not only essentially simplified, but it also allows a geometric interpretation that is wholly analogous to the interpretation of the classical theory in Euclidean geometry.⁴⁷ (Varičák 1912, 105)

⁴⁶For analyses of the exchange, see Jammer (1979), and Miller (1981, 245). Several years later, Varičák offered an alternative explanation of the irreality of the contraction, based on non-Euclidean geometry (Varičák 1924, 77).

⁴⁷"Das Resultat meiner Untersuchung läßt sich dahin aussprechen, daß, unter Zugrundelegung der nichteuklidischen Terminologie, die Formeln der Relativitätstheorie nicht nur wesentlich vereinfacht werden, sondern daß

The non-Euclidean style, in other words, was the one most appropriate to the theory of relativity.

A similar claim had been made on behalf of the space-time formalism by Minkowski and Sommerfeld, as we saw earlier. However, where Minkowski and Sommerfeld accompanied this claim with a display of new physical relations, Varičak arrived empty-handed. He effectively promoted the cause of non-Euclidean geometry in relativity by showing how to express relativistic formulae with hyperbolic functions, and yet he did not offer any new physical insights.

9 Wilson and Lewis's vector calculus

Until 1912, the non-Euclidean style lacked a vector calculus, and thus did not represent a full-fledged alternative to the space-time formalism. Then Edwin Bidwell Wilson (1879–1964), J. Willard Gibbs' last doctoral student, and a professor of mathematics at M. I. T., teamed up with his colleague, the physical chemist Gilbert Newton Lewis (1875–1946), to fill in the gap.

As mentioned above, Lewis had already published a space-time calculus in 1910 (Lewis 1910a, 1910b). The latter work differed from Sommerfeld's formalism in its employment of Gibbs's system of symbolic notation; otherwise, the calculi of both Lewis and Sommerfeld integrated Minkowski's imaginary temporal coordinate. Despite Einstein's praise of his achievement in reformulating Minkowski's four-dimensional matrix calculus, Lewis was not fully satisfied with the reception of his work.⁴⁸ Lewis's system proved to be less popular in Germany than that of Sommerfeld, just as he had predicted.⁴⁹

In his collaboration with Wilson, Lewis kept the same symbolic notation as before. The new approach adopted the non-Euclidean style, by renouncing the use of an imaginary coordinate and introducing in its place an elaborate set of calculation rules. Wilson and Lewis called their 120-page opus "The Space-Time Manifold of Relativity: The Non-Euclidean Geometry of Mechanics and Electromagnetics," and published it in the *Proceedings of the American Academy of Arts and Sciences*, where Lewis's previous work had also appeared.

The new vector calculus, so the authors claimed, challenged Poincaré's "dogmatic" assertion that Euclidean geometry would forever remain the most convenient one for physics (Wilson & Lewis 1912, 329). The limited circulation of the *Proceedings*, however, precluded any such sea-change in theoretical practice. Since Lewis's paper had been translated for publication in Johannes Stark's *Jahrbuch der Radioaktivität und Elektronik*, the M.I.T. pair assumed Stark would also welcome their non-Euclidean paper. However, Stark promptly declined the opportunity to publish their article in German, thereby destroying whatever chance their non-Euclidean formalism might have had to challenge the dominant position of the space-time formalism in relativity theory.⁵⁰

Wilson and Lewis used their non-Euclidean calculus to reproduce Minkowski's fundamental equations, and offered a new derivation of known expressions for the field of an electron in motion. However, they joined in the criticism of Minkowski's definition of ponderomotive force density launched by Einstein, Abraham and others (see above, §4), and described Min-

sie auch eine geometrische Deutung zulassen, die ganz analog ist der Interpretation der klassischen Theorie in der euklidischen Geometrie."

⁴⁸In a letter of December 19, 1910, to the Aachen physicist Johannes Stark (1874–1957), Lewis remarked that Einstein found his system to be the "only logical solution of the 4-dimensional analysis" (Nachlass Stark, Staatsbibliothek Preussischer Kulturbesitz, Berlin).

⁴⁹G. N. Lewis to Arnold Sommerfeld, December 12, 1910, Nachlass Sommerfeld, Deutsches Museum, Munich.

⁵⁰E. B. Wilson to Johannes Stark, October 11, 1912 and November 11, 1912, Nachlass Stark, Staatsbibliothek Preussischer Kulturbesitz.

kowski's appendix on Lorentz-covariant mechanics to be not only "hastily written," but also "fundamentally erroneous." Their target was Minkowski's definition of rest mass density as $\mu\sqrt{1-v^2}$, which is analogous to the formula for rest mass of a material particle; Wilson and Lewis argued that since units of mass and length vary with a change of axes, the correct definition should be $\mu(1-v^2)$ (Wilson & Lewis 1912, 495). The criticism was exaggerated, since Minkowski's definition leads to a correct expression for rest mass, but in making it, Wilson and Lewis implied that their approach was more rigorous than that of Minkowski.

The collaboration of Wilson and Lewis ended with their non-Euclidean calculus, as Lewis left M.I.T. to head the chemistry department at the University of California in Berkeley. Neither of the two took their calculus further, but in a paper written with his student Elliot Q. Adams (1888–1971), Lewis applied the non-Euclidean style to derive formulae of relativistic gas dynamics. Lewis and Adams acknowledged that equivalent relations had been obtained four years earlier by the Breslau mathematician Ferencz Jüttner (1878–1958). While pursuing post-doctoral study in Berlin under the patronage of Max Planck, Jüttner had derived the relativistic modification of the Maxwell distribution law for molecules of a perfect gas. His approach recalled Planck's generalized dynamics, based on the Lorentz-covariant transformation of three-dimensional momentum components, yet Jüttner claimed in a footnote that a more succinct derivation could be obtained with hyperbolic functions pertaining to the four-dimensional space introduced by Minkowski (Jüttner 1911, 873). Lewis and Adams verified this claim.

We have seen that after failing to place their article in a German research journal, Wilson and Lewis abandoned their calculus. The fact that no one rushed to adopt their method reflects the poor diffusion of their work, but this negligence may also be due in part to Wilson and Lewis's failure to demonstrate any practical advantage of their method over the space-time formalism, or to produce any novel empirical or theoretical results. Others working in the non-Euclidean style did no better, except for a Minkowskian mathematician in Paris, Émile Borel.

10 Émile Borel's kinematic space

A former doctoral student of Poincaré, Émile Borel (1871–1956) was renowned for his work on the theory of functions, in which a chair was created for him at the Sorbonne in 1909 (Borel 1912). In the years following his appointment he took up the study of relativity theory, as he said, "in the form given by the late Minkowski."⁵¹ His investigation led to two important insights, communicated both to the Paris Academy of Science and to the students attending his Sorbonne lectures.

Borel's first insight was to identify the geometry of velocity space (or "kinematic" space, in Borel's terminology). In kinematic space, Borel fixed the "defective" assertion that the orientation of the relative velocity of a point with respect to two inertial systems is non-commutative. His version of velocity composition actually involves a significant modification of Einstein's statement of the problem, since it introduces a third inertial observer.⁵² With the fourth data point provided by this observer, Borel could construct a tetrahedron in kinematic space, and determine thereby both the direction and magnitude of relative velocity in a symmetric manner.

⁵¹Borel met Minkowski at the Paris Mathematician's Congress in 1900, and they exchanged correspondence concerning the so-called Borel-Lebesgue theorem (cf. Minkowski to Borel, December 2, 1900, Borel Papers, Bibliothèque de l'Institut Henri Poincaré).

⁵²Einstein's exposé of velocity composition for two inertial systems emphasizes the lack of symmetry in the formula for the direction of the relative velocity vector, see Einstein (1905, 905–906).

No sooner had Borel done this, than a physicist at the Collège de France, Paul Langevin (1872–1946) informed him of Sommerfeld’s priority for the trigonometric demonstration of Einstein’s velocity addition theorem, which Borel acknowledged in his communication to the Paris Academy of Science (Borel 1913b). However, as we saw above (§5), Sommerfeld invoked circular functions without mentioning non-Euclidean geometry; Borel’s acknowledgment of his work prompted a claim from Varičak for the priority of his use of hyperbolic geometry in the study of relativistic kinematics. Borel granted this in a second note to the Paris Academy, and observed on the same occasion that Robb, too, had preceded him in the application of non-Euclidean geometry to relativity (Borel 1913a).

According to Borel, the advantage to be gained in considering velocity addition with respect to kinematic space was partly linguistic, but above all notational. Correct use of this notation by others, however, could not be taken for granted, and soon Borel was prompted to take disciplinary action. Noting with pleasure the Japanese mathematician Kimosuke Ogura’s adoption of the term “kinematic space” (Ogura 1913), Borel deplored the latter’s presentation of the law of velocity addition in its original, non-commutative form. Apparently, Ogura had “not seen all the advantages” of the symmetric form of the law adopted by Borel (Borel 1913a, note 4).

During the course of his study of kinematic space Borel found something “rather curious”: a system of reference whose accelerations are rectilinear for comoving observers may appear to rotate with respect to inertial observers.⁵³ To explain this unusual state of affairs, he recalled that a vector transported parallel to itself along a closed path on the surface of a sphere undergoes a change in orientation at the origin proportional to the enclosed area. In the pseudo-spherical representation of kinematic space, Borel remarked, the same phenomenon occurs: if a system’s point-velocity describes a closed path in kinematic space such that its axes remain stationary for comoving observers, the magnitude of the precession, viewed from a system whose velocity is constant and equal to the initial (and final) velocity of the accelerating system, is equal to the enclosed area. For a circular orbit of radius R and velocity ω , Borel estimated the precession per orbit to be on the order of $R^2\omega^2/c^2$, with an approximate rate of $R^2\omega^3/c^2$.⁵⁴ He was careful to point out that the effect is a direct consequence of the structure of the Lorentz transformations.⁵⁵

Besides this formal argument concerning the orientation of accelerating frames, Borel also predicted the discovery of a physical vector showing a relativistic precession. He surmised that the latter would be detected only in the case of very rapid, periodic particle motion, and provided the example of an orbital radius of 10^{-12} cm and velocity of 3×10^{15} revolutions per second, for which the precession rate is thirty revolutions per second.⁵⁶ Borel pointed out that the possibility of this physical precession opened up a new theoretical vista, since the problem of a rotating solid in the theory of relativity could now be approached from the point of view of the motion of its composite particles.⁵⁷

⁵³Borel 1913b, 215, 217. In modern terms, Borel referred to a “nonrotating” accelerated system, i.e., one whose space vectors do not rotate.

⁵⁴The notation is modified for ease of comparison. Borel chose units in which the velocity of propagation of light was unity, and noted the neglect of a factor of 2π .

⁵⁵Borel elaborated this structure in his Sorbonne lectures (Borel 1914, 42–50).

⁵⁶Units are converted for ease of comparison. Borel did not specify the direction of the precession, and his example implies erroneously that it has the same direction as the orbital motion with respect to the laboratory frame.

⁵⁷Borel 1913b. A somewhat similar view was expressed by Max Born (1910, 234), in defense of his rigid-body definition. The idea of studying rotating bodies from the standpoint of particle precession resurfaced with D. H. Weinstein (1971), according to whom the metric of the rotating disk would be nonstatic due to Thomas

Borel had discovered the kinematic basis for what is known today as Thomas precession, as John Stachel recently pointed out (Stachel 1995, 278). However, as far as the effectiveness of the non-Euclidean style is concerned, the discovery was of limited value, since Borel's effect had a most uncertain physical status.

The non-Euclidean approach Borel used to isolate his effect had to face strong competition. The same year, two young mathematicians in Göttingen derived a precession similar to Borel's—but with greater precision—using the space-time formalism. Ludwig Föppl (1887–1976) and Percy John Daniell (1889–1946) calculated an exact expression for the precession Ω of the axes of a Born-rigid electron in uniform circular orbit, $\Omega = 2\pi(1 - \gamma)$, where $\gamma = 1/\sqrt{1 - v^2/c^2}$. It seems they were unaware of Borel's work, and unlike Borel, they did not ascribe any physical significance to their result. Neither work seems to have attracted much attention, although in one of his notebooks, Einstein graphically illustrated the precession described in an analytic fashion by Föppl and Daniell.⁵⁸

11 Diffusion of the non-Euclidean style

Although the non-Euclidean style had little to show in the way of a creative power of discovery, it still offered a notational advantage over the space-time formalism in some cases. Widely diffused in German journals and textbooks, exposés of the non-Euclidean style were published in Polish, Russian and French journals of mathematics in the pre-war years.⁵⁹ Hyperbolic-functional notation was quickly adopted by mathematicians and theoretical physicists alike for exposés of the law of velocity addition.

A poll of glosses of the velocity addition formula in the handful of relativity textbooks published before the First World War shows that the non-Euclidean style fared about as well as the space-time approach. Writing the first German textbook on relativity, Max Laue, then a *Privatdozent* in Sommerfeld's institute for theoretical physics in Munich, cited Varičák's work (in the non-Euclidean style), but preferred Sommerfeld's imaginary-angle derivation of the velocity addition theorem, based on Minkowski's space-time formalism. Silberstein's textbook took just the opposite tack, while Cunningham's and Weinstein's treatises both ignored the geometric derivations. A mixed approach was also adopted by Heinrich Liebmann, an assistant professor of applied mathematics at Munich Polytechnic, for the second edition of his book on non-Euclidean geometry (Liebmann 1912, §38).

The non-Euclidean style entered the historical annals precociously, thanks to the Cambridge-trained mathematician Edmund T. Whittaker (1873–1956). In the first edition of his delightfully anachronistic history of aether and electricity, the professor of mathematics at Trinity College, Dublin, and Royal Astronomer of Ireland recounted the still-fresh history of relativity with the aid of hyperbolic functions, although in doing so, he did not observe any relation to non-Euclidean geometry (Whittaker 1910, 442). The latter relation was duly noted in both the French and German versions of the *Encyklopädie der mathematischen Wissenschaften*, in the geometry and physics volumes, respectively.⁶⁰ Physicists and mathematicians of the period

precession of the component molecules.

⁵⁸Föppl and Daniell 1913, 528–529; Einstein, *Scratch Notebook*, p. 66, reproduced in Klein 1993a, 596.

⁵⁹For references, see Varičák's bibliography (Varičák 1924).

⁶⁰Fano & Cartan 1915, 41–43; Pauli 1921, 652. Felix Klein also had a hand in the writing of Pauli's authoritative article; not only did Pauli thank Klein for the interest shown in his article, he claimed that the non-Euclidean approach to the Lorentz transformations follows immediately from “well-known arguments by Klein,” when one considers the four coordinate differentials of Minkowski space-time as homogeneous coordinates of a projective space (Pauli 1921, 626, note 111). Klein's interest in general relativity is discussed in David Rowe's contribution

were thereby provided with condensed syntheses of the non-Euclidean style, which continued to find employment in textbooks on special relativity throughout the century.

12 Concluding remarks

The first years of the twentieth century witnessed the development on several fronts of non-Euclidean and n -dimensional geometries, subjects whose utility for mathematical research had been established for a generation. After discovering the hyperbolic geometry of velocity vectors, Minkowski had every reason to believe that his four-dimensional formalism would be favorably met by his colleagues, who were, as he put it, “particularly well predisposed” to develop the theory of relativity (Minkowski, “Das Relativitätsprinzip”, 1, undated typescript, Cod. Ms. Math. Arch. 60: 3, Niedersächsische Staats- und Universitätsbibliothek, Göttingen). His own experience had shown him that expertise in non-Euclidean and hyperspace geometries found ready application in a geometric interpretation of the Lorentz transformation.

Up until the time of Minkowski’s space-time theory, however, non-Euclidean geometry appeared to be irrelevant to physics, and many physicists had undoubtedly neglected to follow the subject in its more advanced topics, notably in differential geometry, and in the study of differential invariants. Minkowski was probably aware of the relatively rudimentary level of mathematical skills possessed by most physicists, and may have considered that non-Euclidean geometry would stand in the way of the acceptance by physicists of his space-time formalism.

Following the total flop of its debut, Minkowski’s formalism steadily gained terrain from the older methods, and soon became the preferred tool of theorists in relativity. Minkowski himself was not fully responsible for this turn of events. After his death, mathematically-adept physicists turned his matrix calculus into a vector and tensor analysis, which found immediate application in electromagnetic theory, thermodynamics, gas dynamics, quantum theory, kinematics, rigid-body dynamics, and elasticity theory.

The emergence of the space-time formalism gave rise to the development of a competitor, the chief characteristics of which we have tried to set out. While the non-Euclidean style intrigued mathematicians, physicists still doubted that non-Euclidean geometry could play an important role in physics. Mathematicians, however, had been sensitized to the latter possibility by Poincaré’s conventionalist philosophy, to which the non-Euclidean style issued a bold challenge.

Physicists’ lack of interest in the non-Euclidean style had several sources. First of all, from its inception the style met with the powerful opposition of Arnold Sommerfeld. In the second place, no vector calculus in the non-Euclidean style was readily available to physicists. For those few who were able to obtain a copy of Wilson and Lewis’s exposé of vector algebra in the non-Euclidean style, rote memorization of a plethora of sign conventions was necessary before useful work could be done. Furthermore, adapting ordinary vector algebra for use in hyperbolic space was just not feasible, as Varičák himself had to admit (Varičák 1924, 80). Third, and perhaps most debilitating of all, the non-Euclidean style counted only one (unconfirmed) physical effect to its credit by 1916. There was little incentive, in other words, for physicists to adopt the non-Euclidean style.

On all these counts the space-time formalism enjoyed a distinct advantage. Sommerfeld energetically promoted it, synthetic presentations of the method were on the shelf (and easily mastered), and a string of surprising physical predictions flowed from the pens of theorists who adopted it. In particular, we have seen how Föppl and Daniell obtained the exact result that

to the present volume. For the Pauli-Klein correspondence from this period see Hermann (1979).

had escaped Borel. Their application of differential geometry to the physics of world-lines in space-time is only one of several such investigations carried out in the heyday of Minkowskian relativity, including Einstein and Grossmann's generalized theory of relativity (1913). The period from 1908 to the outbreak of the First World War was one of intense activity in relativity theory, which saw the introduction of several rival formal techniques, some of which, like the non-Euclidean style, had only limited success. Yet the non-Euclidean style is one example of a general shift in focus to geometric considerations, which constitutes Minkowski's principal heritage in theoretical physics.

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Post-publication note, appended by the author on 2015-09-30. Credit for noticing Minkowski's appeal to hyperbolic space in 1907 belongs to W. F. Reynolds, of whose contribution I was unaware when drafting this chapter; see W. F. Reynolds, *Hyperbolic geometry on a hyperboloid*, *American Mathematical Monthly* 100, 1993, 442-455.

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